Weak slice regular functions in several variables over a weak slice-cone

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The 2022 Annual Conference on Several Complex Variables
August 20, 2022 at Shanghai University

Cayley-Dickson Algebras

- Denote $A_0 := \mathbb{R}$, $e_0 := 1$.
  - Conjugate: $\forall\, x \in A_0$, $\overline{x} := x$.
- $A_{\ell+1} := A_\ell + A_\ell e_{2^\ell}$ is defined by:
  - Multiplication:

$$(a + be_{2^\ell})(c + de_{2^\ell}) := \left(ac - \overline{d}b\right) + (da - b\overline{c})\, e_{2^\ell}, \qquad \forall\, a, b, c, d \in A_\ell.$$

  - Conjugate:

$$\overline{a + be_{2^\ell}} := \overline{a} - be_{2^\ell}, \qquad \forall\, a, b \in A_\ell.$$

- Denote

$$e_{m+2^\ell} := e_m \cdot e_{2^\ell}, \qquad m = 1, 2, ..., 2^\ell - 1.$$

  - $e_0, e_1, ..., e_{2^\ell - 1}$ is a basis of the real vector space $A_\ell \cong \mathbb{R}^{2^\ell}$.

Cayley-Dickson Algebras

- Complex numbers $\mathbb{C} = A_1$, quaternions $\mathbb{H} = A_2$, octonions $\mathbb{O} = A_3$, sedenions $\mathfrak{S} = A_4$.
- A_ℓ, $\ell \leq 3$, is alternative, i.e.

$$\begin{cases} (aa)b = a(ab), \\ a(bb) = (ab)b, \end{cases} \quad \forall\, a, b \in A_\ell.$$

- A_ℓ, $\ell \geq 4$ is not alternative.
  - Let $a := e_1 - e_{10}$ and $b := e_4 + e_{15}$. By directly calculation,

$$ab = ba = 0, \qquad a^2 = -2.$$

Then $(aa)b \neq a(ab)$ since

$$\begin{cases} (aa)b = -2b \neq 0, \\ a(ab) = a \cdot 0 = 0. \end{cases}$$

Slice structure of Quaternions $\mathbb{H} = A_2$

- Imaginary units of $A_2 = \mathbb{H}$:

$$\begin{aligned}\mathbb{S}_{\mathbb{H}} &:= \left\{ I \in \mathbb{H} : I^2 = -1 \right\} \\ &= \left\{ x_1 e_1 + x_2 e_2 + x_3 e_3 : x_\imath \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1 \right\}.\end{aligned}$$

- A slice of $\mathbb{H}$:

$$\mathbb{C}_I := \mathbb{R} + \mathbb{R}I, \qquad I \in \mathbb{S}_{\mathbb{H}}.$$

- Slice structure of $\mathbb{H}$:

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}_{\mathbb{H}}} \mathbb{C}_I.$$

- For each $\Omega \subset \mathbb{H}$, denote

$$\Omega_I := \Omega \cap \mathbb{C}_I, \qquad \Omega_{\mathbb{R}} := \Omega \cap \mathbb{R}.$$

Classical quaternionic analysis

Classcial quaternionic analysis (Fueter, 1934, Comment. Math. Helv.):

- A function $f : \Omega \in \tau(\mathbb{H}) \to \mathbb{H}$ is called Fueter regular, if it satisfies the Cauchy-Fueter equation, i.e.

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right) f(x_0 + x_1 i + xj + x_3 k) = 0.$$

- q, q^n are not Fueter regular.
- Slice quaternionic analysis: Study a class of functions containing $\sum_{n\in\mathbb{N}} q^n a_n$ which is convengence.

Slice quaternionic analysis

Slice analysis study two classes of functions:

- Weak slice regular functions (Gentili and Struppa, Adv. Math., 2007):
  - Certain functions which are holomorphic in each slice ($\mathbb{C}_I,\ I \in \mathbb{S}$).
  - Denote by $\mathcal{WSR}(\Omega)$, $\Omega \subset \mathbb{H}$.
- Strong slice regular functions (Ghiloni and Perotti, Adv. Math., 2011, for studying slice analysis in alternative algebras case):
  - Induced by a holomorphic stem function.
  - Denote by $\mathcal{SSR}(\Omega)$, $\Omega \subset \mathbb{H}$.
  - Holomorphic stem functions: A kind of vector-valued holomorphic functions in complex analysis.
    - Many properties of $\mathcal{SSR}(\Omega)$ are induced by holomorphic stem functions.
- $\mathcal{SSR}(\Omega) \subset \mathcal{WSR}(\Omega)$.

Slice quaternionic analysis

Representation formula for $\mathcal{WSR}(\Omega)$: To find a stem function for a fixed weak slice regular funcion.

- Case of $\Omega = B(0, R) := \{q \in \mathbb{H} : |q| < R\}$ (Gentili and Struppa, 2007, Adv. Math.).
- Case of $\Omega \subset \mathbb{H}$ being symmetric (Colombo, Gentili, Sabadini, Struppa, 2009, Adv. Math.).
- Case of $\Omega \subset \mathbb{H}$ being non-symmetric (Dou, Ren, Sabadini, accepted by JEMS).
  - The representation formula may not hold when Ω is not symmetric. For example: a weak slice regular extension of $F(z) := \sqrt{z - \frac{J}{2}} : (0, +\infty) + \frac{J}{2} \to \mathbb{R}$, where $J \in \mathbb{S}$.
  - A revised formula, called the path-representation formula, hold for this case.
  - To prove path-representation formula, we need introduce a new topolopy, called the slice topolopy.
- Many result in complex anslysis can be extended to weak slice analysis by (path-)representation formula.

Generalized slice analysis

- Weak slice analysis:
  - In one variable (on non-symmetric domains): Case of Clifford algebra $Cl_{0,\ell}$ (e.g. quaternions $\mathbb{H} = Cl_{0,2}$), octonions $\mathbb{O}$.
  - In several variable: on symmetric domains in the weak slice cone $\mathbb{O}_s^n$ (a subset of the strong slice cone $\mathbb{O}^n$).
- Strong slice analysis:
  - In one/several variable: on symmetric domains in the quadratic cone (strong slice cone) Q_A^n, where A is a real alternative $*$-algebra (e.g. $Cl_{m,\ell}$, $\mathbb{O}$).
- Our work: Extend weak slice analysis in several variables to
  - A case of $\mathbb{R}^{2d}$, which includes real alternative $*$-algebras and Cayley-Dickson algebras (e.g. sedenions $\mathfrak{S}$).
  - Our weak slice regular functions are defined on non-symmetric domains in weak slice cones (a subset of strong slice cones).

Slice topology

- Slice topology:

$$\tau_s(\mathbb{H}) := \{\Omega \subset \mathbb{H} : \Omega_I \in \tau(\mathbb{C}_I),\ \forall\ I \in \mathbb{S}\}.$$

  - We call connected sets in τ_s, slice-connected sets. We also call domains in τ_s, slice-domains, etc.
  - $\tau(\mathbb{H}) \subsetneqq \tau_s(\mathbb{H})$.
    - Example: $\Omega \in \tau_s(\mathbb{H})\backslash\tau(\mathbb{H})$. Here fixed $I \in \mathbb{S}_{\mathbb{H}}$, and define

$$\Omega := \bigcup_{J\in\mathbb{S}} \Omega_J,$$

$$\Omega_J := \begin{cases} \{x + yJ \in \mathbb{C}_J : x^2 + \dfrac{y^2}{\text{dist}(J, \mathbb{C}_I)^2} < 1\}, & J \neq \pm I, \\ \{x + yJ \in \mathbb{C}_J : x^2 + y^2 < 1\}, & J = \pm I. \end{cases}$$

Weak slice regular functions

Definition (Gentili and Struppa, Adv. Math., 2007)

A function $f : \Omega \in \tau_s(\mathbb{H}) \to \mathbb{H}$ is called weak slice regular, if for each $I \in \mathbb{S}_{\mathbb{H}}$, $f_I := f\,|_{\Omega_I}$ is I-holomorphic, i.e. f_I is real differentiable and

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+yI) = 0, \qquad \forall x+yI \in \Omega_I.$$

- It is easy to check that $q^n a_n|_{\mathbb{C}_I}$ is I-holomorphic.
  - It implies that if $\sum_{n\in\mathbb{N}} q^n a_n$ is convergence, then $\sum_{n\in\mathbb{N}} q^n a_n \in \mathcal{WSR}(\Omega)$.

Representation Formula

- $\Omega \subset \mathbb{H}$ is called symmetric, if $\Omega = \widetilde{\Omega}$, where

$$\widetilde{\Omega} := \bigcup_{x+yI\in\Omega} x + y\mathbb{S}.$$

Theorem (Colombo, Gentili, Sabadini, Struppa, Adv. Math., 2009)

(Representation Formula) Let $\Omega \subset \mathbb{H}$ is a symmetric slice-domain, and $f : \Omega \to \mathbb{H}$ be weak slice regular. Then

$$f(x+yI) = (1, I)F(x,y), \qquad \forall x+yI \in \Omega$$

where

$$F(x,y) := \begin{pmatrix} 1 & J_1 \\ 1 & J_2 \end{pmatrix}^{-1} \begin{pmatrix} f(x+yJ_1) \\ f(x+yJ_2) \end{pmatrix}$$

is independent of the choice of $J_1, J_2 \in \mathbb{S}_{\mathbb{H}}$ with $J_1 \neq J_2$. F is called the stem function of f.

- The value of f is decided by two holomorphic functions f_{J_1} and f_{J_2}.

Path-representation Formula

- For any path γ in $\mathbb{C}$ and $J \in \mathbb{S}_{\mathbb{H}}$, define

$$\gamma^J := \mathcal{P}_J \circ \gamma,$$

where $\mathcal{P}_J : \mathbb{C} \to \mathbb{C}_J,\ x + yi \mapsto x + yJ, \forall x, y \in \mathbb{R}$.

Theorem (Dou, Ren, Sabadini, accepted by JEMS)

(Path-reprentation Formula) Let $f : \Omega \in \tau_s(\mathbb{H}) \to \mathbb{H}$ be weak slice regular, γ be a path in $\mathbb{C}$ with $\gamma(0) \in \mathbb{R}$. If there are $J_1, J_2 \in \mathbb{S}_{\mathbb{H}}$ with $J_1 \neq J_2$ and $\gamma^{J_1}, \gamma^{J_2} \subset \Omega$, then

$$f \circ \gamma^I = (1, I)F(\gamma), \qquad \forall\ I \in \mathbb{S}_{\mathbb{H}} \text{ with } \gamma^I \subset \Omega,$$

where

$$F(\gamma) := \begin{pmatrix} 1 & J_1 \\ 1 & J_2 \end{pmatrix}^{-1} \begin{pmatrix} f \circ \gamma^{J_1} \\ f \circ \gamma^{J_2} \end{pmatrix}$$

is independent of choice of $J_1, J_2 \in \mathbb{S}_{\mathbb{H}}$. F is called a path-stem function of f.

Strong slice regular functions

In 2011, Ghiloni and Perotti study $\mathcal{SSR}(\Omega)$, when Ω is symmetric.

- A real finite dimensional alternative algebra A is called a real alternative $*$-algebra, if there is an imaginary unit in A, i.e.

$$\varnothing \neq \mathbb{S}_A := \{I \in A : I^2 = -1\}.$$

- The quadratic cone of A:

$$Q_A := \bigcup_{I \in \mathbb{S}_A} \mathbb{C}_I.$$

- Let Ω be a symmetric domain in Q_A. Then $f : \Omega \to A$ is called strong slice regular if there is a (stem function) $F : \Omega_s \to A^{2\times 1}$ such that

$$f(x+yI) = (1, I)F(x,y), \qquad \forall x+yI \in \Omega$$

and F is holomorphic, i.e.

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \frac{\partial}{\partial y}\right) F = 0.$$

where $\Omega_s := \{(x,y) \in \mathbb{R}^2 : \exists I \in \mathbb{S}_A,\ s.t.\ x+yI \in \Omega\}$.

Strong slice regular functions

- $\mathcal{SSR}(\Omega) = \mathcal{WSR}(\Omega)$ when Ω is a symmetric domain in Q_A.
  - Ghiloni and Perotti prove that the part of $\mathcal{SSR}(\Omega) \subset \mathcal{WSR}(\Omega)$.
  - When A is octonions or Clifford algebra $Cl_{0,n}$.
    - $\mathcal{SSR}(\Omega) \supset \mathcal{WSR}(\Omega)$ holds directly by representation formula.
  - Representation formula does not hold for general case, since $J_1 - J_2$ may not be inverse, so is $\begin{pmatrix} 1 & J_1 \\ 1 & J_2 \end{pmatrix}$.
    - However, representation formula also holds when $J_1 = -J_2$.
    - It also enough to prove that $\mathcal{SSR}(\Omega) \supset \mathcal{WSR}(\Omega)$.
- We would like to study $\mathcal{WSR}(\Omega)$ when Ω be non-symmetric.
  - We can not choice J_1 to be $-J_2$, since Ω is not symmetric.
    - $J_1 - J_2$ may not be inverse.
    - We will use Moore-Penrose inverse.

Strong slice regular functions in several variables

- Ghiloni and Perotti (Math. Z., 2022) study a class of strong slice regular functions $\mathcal{SSR}(\Omega)$ on $Q_A^n := (Q_A)^n$, where A is a real alternative $*$-algebra.
  - In case of $n = 2$, a strong slice regular function $f : \Omega(\subset Q_A^2) \to A$ and its stem function $F : \Omega_s(\subset \mathbb{C}^2) \to A^{2^2 \times 1}$ satisfying

$$f(x_1 + y_1 I, x_2 + y_2 J) = (1, I, J, IJ)F(x_1 + y_1 i, x_2 + y_2 i) \qquad (1)$$

    Here the stem function F is holomorphic, i.e.

$$\begin{cases} \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \sigma_1 \frac{\partial}{\partial y_1}\right) F(x_1 + y_1 i, x_2 + y_2 i) = 0, \\ \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \sigma_2 \frac{\partial}{\partial y_2}\right) F(x_1 + y_1 i, x_2 + y_2 i) = 0, \end{cases}$$

    where

$$\sigma_1 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

- Example: $q_1^{k_1} q_2^{k_2} \cdots q_n^{k_n} b \in \mathcal{SSR}(\Omega)$, where $b \in A$.

Strong slice regular functions in several variables

- The class of strong slice regular functions is not unique, if we do not consider the holomorphy of f on slice $\mathbb{C}_I \times \mathbb{C}_J$.
  - For example, if we replace (1) with

$$f(x_1 + y_1 I, x_2 + y_2 J) = (1, I, J, JI) F(x_1 + y_1 i, x_2 + y_2 i).$$

  And gain a class of strong slice regular functions, denoted by $\mathcal{SSR}_1(\Omega)$. Then

$$q_1 q_2 \in \mathcal{SSR}(\Omega) \backslash \mathcal{SSR}_1(\Omega), \qquad q_2 q_1 \in \mathcal{SSR}_1(\Omega) \backslash \mathcal{SSR}(\Omega).$$

- When $A = \mathbb{O}$, Dou, Ren, Sabadini and Yang (JGA, 2021) study a class of functions which is holomorphic on $\mathbb{C}_I \times \mathbb{C}_I$, $\forall\, I \in \mathbb{S}_A$, i.e. weak slice regular functions $\mathcal{WSR}(\Omega)$, where Ω is a symmetric slice-open set in

$$\mathbb{O}_s^2 = (Q_A)_s^n = \bigcup_{I \in \mathbb{S}_A} \mathbb{C}_I \times \mathbb{C}_I, \qquad \left(Q_A^2 = \bigcup_{I,J \in \mathbb{S}_A} \mathbb{C}_I \times \mathbb{C}_J \right).$$

We call $(Q_A)_s^n$ weak slice cone and Q_A^n strong slice cone.

Strong slice regular functions in several variables

- If $A = \mathbb{O}$ (for general real alternative algebra A is similar) then

$$\mathcal{SSR}(\Omega)|_{(Q_A)_s^n} = \mathcal{WSR}\left(\Omega|_{(Q_A)_s^n}\right).$$

  - Here $\mathcal{SSR}(\Omega)$ and $\mathcal{WSR}(\Omega)$ are both only studied when Ω is symmetric.
- We want to study slice regular functions defined on some non-symmetric set Ω.
  - It is hard to find a 'good' class of 'holomorphic' functions defined on $\mathbb{C}_I \times \mathbb{C}_J \times \mathbb{C}_K$. So we also study weak slice regular functions defined on a slice-open set on weak slice cone, e.g. $(Q_A)_s^n$.

Weak slice cone

- We replace real alternative $*$-algebra A with $\mathbb{R}^{2n}$ and study $\mathcal{WSR}(\Omega)$ in this case, (see Dou, Ren, Sabadini, arXiv:2011.13770).
  - A set of complex structures:
  $$\mathcal{C} \subset \{I \in End\left(\mathbb{R}^{2n}\right) : I^2 = -id_{\mathbb{R}^{2n}}\}.$$
  with $\mathcal{C} = -\mathcal{C}$.
  - Weak slice cone:
  $$\mathcal{W}_{\mathcal{C}}^d := \bigcup_{I \in \mathcal{C}} \mathbb{C}_I^d \subset \left[End\left(\mathbb{R}^{2n}\right)\right]^d.$$
  - Slice topology:
  $$\tau\left(\mathcal{W}_{\mathcal{C}}^d\right) := \left\{\Omega \subset \mathcal{W} : \Omega_I \in \tau\left(\mathbb{C}_I^d\right)\right\}.$$
- Remark: In case of real alternative $*$-algebra A, $\mathcal{C}$ corresponds $\mathbb{S}_A$,and $\mathcal{W}_{\mathcal{C}}^d$ corresponds $(Q_A)_s^n$.

Weak slice cone

- Our theory covers not only the real alternative $*$-algebra case, but also someother algebras called left slice complex structure algebras, LSCS algebras for short.
  - A real finite-dimensional unital algebra $A \neq \{0\}$ is called an LSCS algebra for short, if there is $b \in A$ such that L_b is a complex structure on A.
    - Here $L_b : A \to A, x \mapsto bx$.
  - Certain real left alternative algebras are LSCS algebra.
    - Real alternative $*$-algebras are in this case, which includes Clifford algebras (e.g. complex numbers, quaternions, split-quaternions) and octonions.

  Moreover, Cayley-Dickson algebras $A_\ell, \ell > 0$ are LSCS algebras.

Weak slice cone

- In LSCS algebras case, we set

$$\mathcal{C} = \mathcal{C}_A := \{L_b : b \in A,\ (L_b)^2 = -id_A\}.$$

  - If A is left alternative, then by $L_a L_a = L_{a^2}$,

$$\mathcal{C}_A = \{L_a : a \in A,\ a^2 = -1\} = \{L_a : a \in \mathbb{S}_A\}.$$

  - In case of sedenions $\mathfrak{S}$,

$$\mathcal{C}_{\mathfrak{S}} = \{a + be_8 \in \mathbb{S}_{\mathfrak{S}} : a, b \in \mathbb{O} \text{ with } ab = ba\}$$

- $f : \Omega \in \tau_s\left(\mathcal{W}_{\mathcal{C}}^d\right) \to \mathbb{R}^{2n}$ is called weak slice regular, if

$$\frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + I\frac{\partial}{\partial y_\ell}\right) f(x + yI) = 0, \qquad \forall\, x + yI \in \Omega,$$

$\ell = 1, 2, ..., d.$

Weak slice cone

- By similar method for the case of one quaternionic variable, many results holds:
  - (Splitting Lemma) Let $\Omega \in \tau_s\left(\mathcal{W}_{\mathcal{C}}^d\right)$. $f : \Omega \to \mathbb{R}^{2n}$ is weak slice regular if and only if for any $I \in \mathcal{C}$ and I-basis $\{\xi_1, ..., \xi_n\}$, there are n holomorphic functions $F_1, ..., F_n : \Omega_I \to \mathbb{C}_I$, such that

$$f_I = \sum_{\ell=1}^{n} (F_\ell \xi_\ell).$$

  - (Identity Principle) Let Ω be a slice-domain in $\mathcal{W}_{\mathcal{C}}^d$, and $f, g : \Omega \to \mathbb{R}^{2n}$ be weak slice regular. Then
    - If $\Omega_{\mathbb{R}} \neq \varnothing$ and f, g coincide on a non-empty open subset of $\Omega_{\mathbb{R}}$, then $f = g$ on Ω.
    - If f, g coincide on a non-empty open subset of Ω_I for some $I \in \mathcal{C}$, then $f = g$ on Ω.

  (Also need similar method for the case of $\mathbb{O}_s^n$.)

Moore-Penrose inverse

- For each $A \in End(\mathbb{R}^{2n})^{\ell\times\ell} \cong \mathbb{R}^{2n\ell\times 2n\ell}$, denote by A^* the transpose of $A \in \mathbb{R}^{2n\ell\times 2n\ell}$ as a real matrix.
  - For example, let $n = 1$, $\ell = 2$ and $A = \begin{pmatrix} & I \\ & \end{pmatrix}$ where $I = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Then

$$A^* = \begin{pmatrix} & \\ I^* & \end{pmatrix} = \begin{pmatrix} & \\ -I & \end{pmatrix} \qquad \text{and we denote} \qquad A^T := \begin{pmatrix} & \\ I & \end{pmatrix}.$$

- Let $J \in \mathrm{End}\left(\mathbb{R}^{2n}\right)^{k\times\ell}$. Then there is a unique matrix J^+ in $\mathrm{End}\left(\mathbb{R}^{2n}\right)^{\ell\times k}$ (called the Moore-Penrose inverse of J) that satisfies the Moore-Penrose conditions:
  - $JJ^+J = J, \qquad J^+JJ^+ = J^+.$
  - $(JJ^+)^* = JJ^+, \qquad (J^+J)^* = J^+J.$
- Fix a complex structure $\mathbb{J}$ in $\mathbb{R}^{2n}$ with $\mathbb{J}^*\mathbb{J} = id_{\mathbb{R}^{2n}}$. Then for each complex structure $I \in \mathcal{C}$, we choose a fixed $D_I \in \mathrm{End}(\mathbb{R}^{2n})$ with

$$I = D_I \mathbb{J} (D_I)^{-1}.$$

Extension Lemma

- Let $J = (J_1, ..., J_k) \in \mathcal{C}^k$. Define

$$D_J := \begin{pmatrix} D_{J_1} & & \\ & \ddots & \\ & & D_{J_k} \end{pmatrix}, \qquad \mathrm{diag}(J) := \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

and

$$\zeta(J) := \begin{pmatrix} 1 & J_1 \\ \vdots & \vdots \\ 1 & J_k \end{pmatrix}$$

- We call

$$\zeta^+(J) := [D_J \cdot \zeta(J)]^+ D_J$$

the J-slice inverse of $\zeta(J)$.

- Let $I \in \mathcal{C}$ and $J = (J_1, ..., J_k) \in \mathcal{C}^k$. Then
  - $I[(1, I)\zeta^+(J)] = [(1, I)\zeta^+(J)]\mathrm{diag}(J)$.

Extension Lemma

- For any $\Omega \subset \mathcal{W}_{\mathcal{C}}^d$, define

$$\mathscr{P}(\mathbb{C}^d) := \{\gamma : [0,1] \to \mathbb{C}^d,\ \gamma \text{ is a path s.t. } \gamma(0) \in \mathbb{R}^d\};$$

$$\mathscr{P}\left(\mathbb{C}^d, \Omega\right) := \left\{\delta \in \mathscr{P}\left(\mathbb{C}^d\right) : \exists\, I \in \mathcal{C},\ \text{s.t. } \delta^I \subset \Omega\right\},$$

and for each $\gamma \in \mathscr{P}\left(\mathbb{C}^d\right)$ we define

$$\mathcal{C}(\gamma, \Omega) := \left\{I \in \mathcal{C} : \gamma^I \subset \Omega\right\}.$$

- Let $J = (J_1, ..., J_k) \in \mathcal{C}^k$, $\Omega \subset \mathcal{W}_{\mathcal{C}}^d$ and $\gamma \in \mathscr{P}(\mathbb{C}^d, \Omega)$. We define

$$\mathcal{C}_{ker}(J) := \left\{I \in \mathcal{C} : \ker(1, I) \supset \bigcap_{\ell=1}^{k} \ker(1, J_\ell)\right\},$$

and

$$\mathcal{C}(\Omega, \gamma, J) := \mathcal{C}(\Omega, \gamma) \cap \mathcal{C}_{ker}(J).$$

Extension Lemma

Lemma

(Extension Lemma) Let $U \in \tau(\mathbb{C}^d)$, $I \in \mathcal{C}$ *and* $J = (J_1, ..., J_k) \in \mathcal{C}^k$. *If* $g_\ell : U^{J_\ell} \to \mathbb{R}^{2n}$, $\ell = 1, ..., k$ *are holomorphic, then* $g[I] : U^I \to \mathbb{R}^{2n}$ *defined by*

$$g[I](x + yI) = (1, I)\zeta^+(J)g(x + yJ), \qquad \forall\, x + yi \in U,$$

where

$$g(x + yJ) = \begin{pmatrix} g_1(x + yJ_1) \\ \vdots \\ g_k(x + yJ_k) \end{pmatrix}$$

is holomorphic.
Moreover, if $U_{\mathbb{R}} := U \cap \mathbb{R}^d \neq \varnothing$, $g_1 = \cdots = g_k$ *on* $U_{\mathbb{R}}$ *and* $I \in \mathcal{C}_{ker}(J)$, *then*

$$g[I] = g_1 = \cdots = g_k \qquad \text{on} \qquad U_{\mathbb{R}}.$$

Key part of the proof.

For each $\ell \in \{1, ..., d\}$ and $x + yi \in U$,

$$\frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + I\frac{\partial}{\partial y_\ell}\right) g[I](x + yI)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + I\frac{\partial}{\partial y_\ell}\right) (1, I)\zeta^+(J) g(x + yJ)$$

$$= (1, I)\zeta^+(J) \begin{pmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + J_1\frac{\partial}{\partial y_\ell}\right) & & \\ & \ddots & \\ & & \frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + J_k\frac{\partial}{\partial y_\ell}\right) \end{pmatrix} \begin{pmatrix} g_1(x + yJ_1) \\ \vdots \\ g_k(x + yJ_k) \end{pmatrix}$$

$$= (1, I)\zeta^+(J) \begin{pmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + J_1\frac{\partial}{\partial y_\ell}\right) g_1(x + yJ_1) \\ \vdots \\ \frac{1}{2}\left(\frac{\partial}{\partial x_\ell} + J_k\frac{\partial}{\partial y_\ell}\right) g_k(x + yJ_k) \end{pmatrix} = 0.$$

Hence $g[I]$ is holomorphic. □

Path-representation Formula

By similar method for the case of one quaternionic variable, we have

Theorem (Main theorem)

(Path-representation Formula) Let $\Omega \in \tau_s(\mathcal{W}_{\mathcal{C}}^d)$, $\gamma \in \mathscr{P}(\mathbb{C}^d, \Omega)$, $J = (J_1, J_2, ..., J_k) \in [\mathcal{C}(\gamma, \Omega)]^k$ *and* $I \in \mathcal{C}(\gamma, \Omega, J)$. *If* $f : \Omega \to \mathbb{R}^{2n}$ *is weak slice regular, then*

$$f \circ \gamma^I = (1, I) F(\gamma, J),$$

where

$$F(\gamma, J) = \zeta^+(J)(f \circ \gamma^J), \qquad \text{and} \qquad f \circ \gamma^J := \begin{pmatrix} f \circ \gamma^{J_1} \\ \vdots \\ f \circ \gamma^{J_k} \end{pmatrix}.$$

Hyper-sigma-ball

- In quaternionic case, let $I \in \mathbb{S}_{\mathbb{H}}$ and $a \in \mathbb{H}$, then the domain of convergence of the series

$$f(q) = \left[\sum_{n\in\mathbb{N}} \left(q - \frac{I}{2}\right)^{*2^n}\right] a,$$

is the σ-ball

$$\Sigma\left(\frac{I}{2}, 1\right) = \left\{\bigcup_{J\in\mathbb{S}_{\mathbb{H}}} \mathcal{P}_J \left[B\left(\frac{i}{2}, 1\right) \cap B\left(-\frac{i}{2}, 1\right)\right]\right\} \bigcup \left\{\mathcal{P}_I \left[B\left(\frac{i}{2}, 1\right)\right]\right\}.$$

- However, in sedenionic case, let $I = e_1$ and $a = e_4 + e_{15}$, then the domain of convergence is the hyper-σ-ball:

$$\left\{\bigcup_{J\in\mathbb{S}_{\mathfrak{S}}} \mathcal{P}_J \left[B\left(\frac{i}{2}, 1\right) \cap B\left(-\frac{i}{2}, 1\right)\right]\right\} \bigcup \left\{\bigcup_{K\in\mathcal{S}} \mathcal{P}_K \left[B\left(\frac{i}{2}, 1\right)\right]\right\},$$

where $\mathcal{S} = \{(\cos\theta e_1 + \sin\theta e_2)(\cos\theta + \sin\theta e_8) : \theta \in [0, \pi)\}$.

Thanks