Holomorphic Vector Bundles on Homogeneous spaces

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Outline

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Everything is over $\mathbb{C}$, E: holomorphic r-bundle.

Theorem (Grothendieck 1956)

Every r-bundle E over $\mathbb{P}^1$ (splits) has the form

$$E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

with uniquely determined numbers $a_1, \ldots, a_r \in \mathbb{Z}$,
$a_1 \geq a_2 \geq \cdots \geq a_r$.

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Question

How about r-bundle over $\mathbb{P}^n$, $n \geq 2$?

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4. rank $r \leq n - 2$: is hard.

Vector bundles on $\mathbb{P}^n$

1. 1973, Horrocks and Mumford: $n = 4$, rank $r = 2$.

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$Y = \mathbb{C}^2/\Gamma$.

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2. 1978, Horrocks: $n = 5$, rank $r = 3$.

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Any rank 2 vector bundle over $\mathbb{P}^n$ $(n \geq 7)$ splits into the direct sum of two line bundles.

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$n = 5$, $n = 6$, $r = 2$ still open!

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$$a_E : G(2, n+1) \to \mathbb{Z}^r, \;\; l \mapsto a_E(l) = (a_1(l), \cdots, a_r(l))$$

where $E|l \cong \bigoplus_{i=1}^{r} \mathcal{O}_L(a_i(l))$, $a_1(l) \geq \cdots \geq a_r(l)$.

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Definition

$a_E(l)$: splitting type of E on l.
E is called uniform if a_E is constant.

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$$Ext^1_H(\mathcal{O}_H(1), T_H) = 0$$

$$\Rightarrow T_{\mathbb{P}^n}|H = T_H \oplus \mathcal{O}_H(1).$$

By induction and $T_L = \omega_L^* = \mathcal{O}_L(2)$.

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$T_{\mathbb{P}^n}$ is uniform of splitting type $a_{T_{\mathbb{P}^n}} = (2, 1, \cdots, 1)$.

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5. 1980, Sato+Elencwajg : complete classify uniform 3-bundles over $\mathbb{P}^n$.

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2. 1982, Ellia, Ballico independently: uniform $(n+1)$-bundles $(n \geq 3)$ over $\mathbb{P}^n$:

$$\oplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^n}(a_i),\ T_{\mathbb{P}^n}(a)\oplus\mathcal{O}_{\mathbb{P}^n}(b) \quad \text{or} \quad \Omega^1_{\mathbb{P}^n}(c)\oplus\mathcal{O}_{\mathbb{P}^n}(d).$$

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3. 1985, Guyot: uniform d-bundle on Grassmannian $G = G(d,n)$:

$$\oplus_{i=1}^{d} \mathcal{O}_G(a_i),\ H_d(a), H_d^{\vee}(b),$$

where H_d is the tautological sub-bundle.

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Question (Muñoz-Occhetta-Solá Conde)

Classify low rank uniform principal G-bundles (G semisimple algebraic group) on rational homogeneous spaces.

Lie group, Lie algebra and rational homogeneous space

Rational homogeneous space:

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$G_1, \cdots, G_m$: simple algebraic groups.
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G_i/P_{I_i}: generalized flag manifold.

Semi-simple Lie group and Lie algebra

Lie group-Lie algebra correspondence allows one to study Lie groups, which are geometric objects, in terms of Lie algebras, which are linear objects.

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$H \subset G$: a maximal torus $\rightsquigarrow$ $\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra (abelian subalgebra of maximal dimension)

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$H \subset G$: a maximal torus $\rightsquigarrow$ $\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra (abelian subalgebra of maximal dimension)
Cartan decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} | ad_\mathfrak{g}(h)(g) = \alpha(h)g, \text{ for all } h \in \mathfrak{h}\}$$

$$ad_\mathfrak{g} : \text{adjoint representation}$$

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$$f : \mathrm{span}_{\mathbb{R}} \Phi \to \mathbb{R}$$

whose kernel does not intersect Φ. Let

$$\Phi^+ := \{\alpha \in \Phi | f(\alpha) > 0\} \text{ and } \Phi^- := \{\alpha \in \Phi | f(\alpha) < 0\}.$$

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simple system $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi^+$: $\alpha \in \Pi \Leftrightarrow \alpha \in \Phi^+$ and α cannot be expressed as the sum of two elements of Φ^+.

Semi-simple Lie group and Lie algebra

Killing form: $(\alpha, \beta) := tr(\mathrm{ad}_\alpha \circ \mathrm{ad}_\beta)$ defines a nondegenerated bilinear form on $\mathfrak{h}$, where $\alpha, \beta \in \mathfrak{g}$.

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Killing form: $(\alpha, \beta) := tr(\mathrm{ad}_\alpha \circ \mathrm{ad}_\beta)$ defines a nondegenerated bilinear form on $\mathfrak{h}$, where $\alpha, \beta \in \mathfrak{g}$.
Cartan matrix: $A = (A_{ij})$,

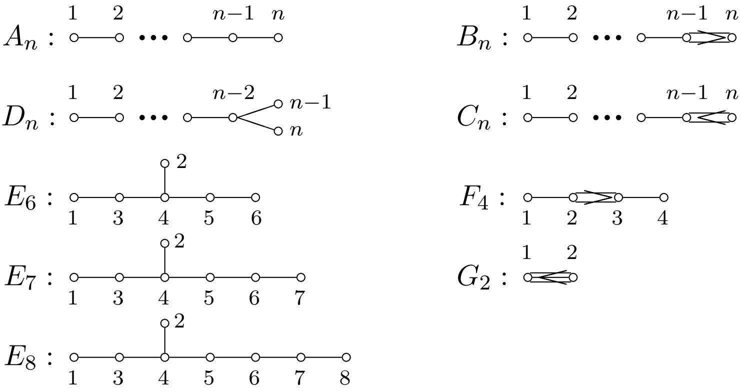
$$A_{ij} :=< \alpha_i, \alpha_j >= 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad i, j = 1, \ldots n.$$

Semi-simple Lie group and Lie algebra

Dynkin diagram:

- nodes $\leftrightarrow \Pi = \{\alpha_1, \ldots, \alpha_n\}$
- $\sharp$ edges connecting α_i and $\alpha_j = A_{ij}A_{ji}$
- arrow for double or triple edges from α_j to α_i if $|A_{ij}| > |A_{ji}|$ (α_i: short root; α_j: long root)

Dynkin diagrams



Parabolic subgroup and subalgebra

$B \leq G$: Borel subgroup (maximal connected solvable subgroup)

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So we use notation $\mathfrak{p}_I$ and P_I.

Rational homogeneous spaces

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The two extremal cases: *generalized complete flag manifolds* (all nodes marked), *generalized Grassmannians* (only one node marked).

Lines on rational homogeneous spaces

Theorem (Landsberg-Manivel 2003)

Let $I \subseteq D = \{1, \ldots, n\}$. Suppose G to be a simple Lie group. Consider $X = G/P_I$ in its minimal homogeneous embedding. Denote by $F_1(X)$ the space of $\mathbb{P}^1$'s in X. Then

1. $F_1(X) = \coprod_{j \in I} F_1^j(X)$, *where $F_1^j(X)$ is the space of lines of class $\check{\alpha}_j \in H_2(X, \mathbb{Z})$.*
2. *If α_j is not an exposed short root, then* $F_1^j(X) = G/P_{(I \setminus j) \cup N(j)}$.
3. *If α_j is an exposed short root, then $F_1^j(X)$ is the union of two G-orbits, an open orbit and its boundary $G/P_{(I \setminus j) \cup N(j)}$ (called special family of lines).*

Lines on rational homogeneous spaces

Definition

We call $\alpha_j (j \in I)$ an exposed short root if the connected component of j in $D \backslash (I \backslash j)$ contains root longer than α_j, i.e., if an arrow in $D \backslash (I \backslash j)$ points towards j.

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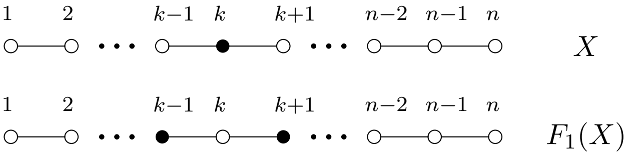
Example

A_n, i.e. $X = SL_{n+1}/P_I$ is the *generalized flag manifold*.
For $I = \{k\}$, X is the usual Grassmannian and $F_1(X)$ is just the variety of lines on X.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$F_1(X)$

Uniform vector bundles on rational homogeneous spaces

Theorem (D-Gao-Fang 2021)

Suppose that E is a uniform r-bundle of a generalized Grassmann $\mathcal{G}$ with respect to the special family of lines. If $r \leq \varsigma(\mathcal{G})$, then E splits as a direct sum of line bundles.

Table: $\varsigma(\mathcal{G})$

$\mathcal{G}$	A_n/P_1 A_n/P_n	B_n/P_1 B_n/P_n	C_n/P_1 C_n/P_n	D_n/P_1 D_n/P_{n-1} D_n/P_n	E_6/P_1 E_6/P_2 E_6/P_6	E_7/P_1 E_7/P_2 E_7/P_7	E_8/P_1 E_8/P_2 E_8/P_8	F_4/P_1 F_4/P_4	G_2/P_1 G_2/P_2
$\varsigma(\mathcal{G})$	$n-1$ $n-1$	$2n-3$ $n-1$	$2n-3$ $n-1$	$2n-5$ $n-1$ $n-1$	7 5 7	9 6 12	11 7 14	5 5	1 1

Uniform vector bundles on rational homogeneous spaces

Consider rational homogeneous space $X = G/P$. If $\delta_i \in I_i$, we call

$$\mathcal{M}_i^{\delta_i^c} := G_i/P_i^{\delta_i^c} \times \widehat{G_i/P_{I_i}} \ (1 \leq i \leq m),$$

the i*-th special family* of lines, where $P_i^{\delta_i^c} := P_{(I_i \backslash \delta_i) \cup N(\delta_i)}$ and $\widehat{G_i/P_{I_i}}$ is $G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m}$ by deleting i-th term G_i/P_{I_i}.

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Definition

A vector bundle E on X is called poly-uniform with respect to $\mathcal{M}_i^{\delta_i^c}$ for every i $(1 \leq i \leq m)$ and every $\delta_i \in I_i$ if the restriction of E to every line in $\mathcal{M}_i^{\delta_i^c}$ has the same splitting type. We also call that E poly-uniform with respect to all the special families of lines.

δ_i-slope

Let $\mathscr{F}$ be a torsion free coherent sheaf of rank r over X. Fix integer i $(1 \leq i \leq m)$ and $\delta_i \in I_i$.

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$$\mathscr{F}|L \cong \mathcal{O}_L(a_1^{(\delta_i)}) \oplus \cdots \oplus \mathcal{O}_L(a_r^{(\delta_i)}),$$

we set

$$c_1^{(\delta_i)}(\mathscr{F}) = a_1^{(\delta_i)} + \cdots + a_r^{(\delta_i)}$$

and

$$\mu^{(\delta_i)}(\mathscr{F}) = \frac{c_1^{(\delta_i)}(\mathscr{F})}{\text{rk}(\mathscr{F})},$$

which are independent of the choice of L.

δ_i-semistable

Definition

A torsion free coherent sheaf $\mathscr{E}$ over X is δ_i-semistable if for every coherent subsheaf $\mathscr{F} \subseteq \mathcal{E}$, we have

$$\mu^{(\delta_i)}(\mathscr{F}) \leq \mu^{(\delta_i)}(\mathcal{E}).$$

If E is not δ_i-semistable, then we call E is δ_i-unstable.

Uniform vector bundles on rational homogeneous spaces

Denote $\nu(X,\delta_i) := \varsigma(\mathcal{G}^{\delta_i})$, where $\varsigma(\mathcal{G}^{\delta_i})$ are defined as in Table. Let

$$\nu(X) := \text{min}_i\{\text{min}_{\delta_i \in I_i}\{\nu(X,\delta_i)\}\}.$$

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Theorem (D-Gao-Fang 2021)

On X, if an r-bundle E is poly-uniform with respect to all the special families of lines and $r \leq \nu(X)$, then E is δ_i-unstable for some δ_i $(1 \leq i \leq m)$ or E splits as a direct sum of line bundles.

Grauert-Mülich theorem on $\mathbb{P}^n$

$a_E : G = G(2, n+1) \to \mathbb{Z}^r$, $a_E(l) = (a_1(l), \cdots, a_r(l))$,
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Give $\mathbb{Z}^r$ lexicographical ordering: $(a_1, \cdots, a_r) \leq (b_1, \cdots, b_r)$ if the first non-zero difference $b_i - a_i$ is positive.

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Let

$$\underline{a}_E = \inf_{l \in G} a_E(l)$$

Definition

$S_E = \{l \in G | a_E(l) > \underline{a}_E\}$ is the set of jumping lines. $\underline{a}_E$ is the generic splitting type of E.

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Remark: $U_E = G \setminus S_E$ is a non-empty Zariski open subset of G.

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Theorem (Grauert-Mülich 1977)

Let E be a semistable 2-bundle over $\mathbb{P}^n$ and the generic splitting type of E is (a_1, a_2). Then $a_1 - a_2 = 0$ or 1.

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Theorem (Splindler 1979)

Let E be a semistable r-bundle over $\mathbb{P}^n$. Then $a_i - a_{i+1} = 0$ or 1, $i = 1, \ldots, r-1$.

Generalized Grauert-Mülich theorem

Theorem (Guyot 1985)

Let E be a semistable r-bundle over Grassmannian and the generic splitting type of E is $(a_1, \cdots, a_r)$. Then $a_i - a_{i+1} = 0$ or 1, $i = 1, \ldots, r-1$.

Generalized Grauert-Mülich theorem

Theorem (D-Gao-Fang 2021)

Fix $\delta_i \in I_i$. Let E be an r-bundle over X of general splitting type $\underline{a}_E^{(\delta_i)} = (a_1^{(\delta_i)}, \dots, a_r^{(\delta_i)})$, where $a_1^{(\delta_i)} \geq \cdots \geq a_r^{(\delta_i)}$, with respect to $\mathcal{M}_i^{\delta_i^c}$. For a δ_i-semistable r-bundle E over X of general splitting type $\underline{a}_E^{(\delta_i)} = (a_1^{(\delta_i)}, \dots, a_r^{(\delta_i)})$, where $a_1^{(\delta_i)} \geq \cdots \geq a_r^{(\delta_i)}$, with respect to $\mathcal{M}_i^{\delta_i^c}$ and all $s = 1, \dots, r-1$, we have

$$a_s^{(\delta_i)} - a_{s+1}^{(\delta_i)} \leq \begin{cases} 1, & \text{if } \delta_i \text{ is not an exposed short root;} \\ 2, & \text{if } \delta_i \text{ is an exposed short root and } \delta_i \notin \mathcal{D}(G_2); \\ 3, & \text{if } \delta_i \text{ is an exposed short root and } \delta_i \in \mathcal{D}(G_2). \end{cases}$$

Homogeneous vector bundles on homogeneous spaces

Homogeneous vector bundles on $\mathbb{P}^n$:

Definition

An r-bundle E over $\mathbb{P}^n$ is homogeneous if for every projective transformation $t \in PGL(n+1, \mathbb{C})$, we have $t^*E = E$.

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Conjecture

Every uniform vector bundle of rank $r < 2n$ is homogeneous.

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Conjecture

Every rank $r < 2n$ uniform vector bundle on $\mathbb{P}^n$ is a direct sum of bundles chosen among:

$$S^2 T_{\mathbb{P}^2}(a), \wedge^2 T_{\mathbb{P}^4}(b), T_{\mathbb{P}^n}(c), \Omega_{\mathbb{P}^n}(d), \mathcal{O}_{\mathbb{P}^n}(e).$$

Homogeneous vector bundles on homogeneous spaces

G/P: rational homogeneous variety, G simply connected and semi-simple group, P parabolic subgroup

Definition

Over G/P, a vector bundle E is called *homogeneous* if there exists an action G over E such that the following diagram commutes

$$\begin{array}{ccc} G \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \times G/P & \longrightarrow & G/P. \end{array}$$

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Definition

A vector bundle E over G/P is homogeneous $\Leftrightarrow$

$$\theta_g^* E = E, \forall \theta_g \in Aut(G/P), g \in G.$$

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Let $\rho : P \to GL(r)$ be a representation. In $G \times \mathbb{C}^r$,

$$(g, v) \sim (g', v') \Leftrightarrow \exists p \in P \text{ such that } g = g'p \text{ and } v = \rho(p^{-1})v'.$$

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Theorem (Matsushima)

A r-bundle E over G/P is homogeneous $\Leftrightarrow$ there exists a representation $\rho : P \to GL(r)$ such that $E \simeq E_\rho$.

Homogeneous vector bundles on homogeneous spaces

A weight λ of G: linear function $\lambda : \mathfrak{h} \to \mathbb{C}$ such that $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$.

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Proposition

Let $I = \{\alpha_1, \dots, \alpha_k\}$ be a subset of simple roots. Let $\lambda_1, \dots, \lambda_k$ be the corresponding fundamental weights. Then all the irreducible representations of P_I are

$$V \otimes L_{\lambda_1}^{n_1} \otimes \cdots \otimes L_{\lambda_k}^{n_k},$$

where V is a representation of S_P (the semisimple part of P), $n_i \in \mathbb{Z}$ and L_{λ_i} is a one-dimensional representation with weight λ_i.

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So the irreducible representation of P_I is determined by its highest weight.

Arithmetically Cohen-Macaulay bundle

Definition

A vector bundle E on a smooth projective variety X is called arithmetically Cohen-Macaulay (ACM) if $H^i(X, E(t))$ for $0 < i <$ dim$X$ and all $t \in \mathbb{Z}$.

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ACM bundles on $\mathbb{P}^n$ split.

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2016, Costa and Miró-Roig: classify the irreducible homogeneous ACM bundles on Grassmannians.

ACM bundle on isotropic Grassmannians

Definition

Given a projective variety $(X, \mathcal{O}_X(1))$, a vector bundle E on X is called *initialized* if

$$H^0(X, E(-1)) = 0$$

and

$$H^0(X, E) \neq 0.$$

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Theorem (D-Fang-Ren, ≥ 2022)

Let E_λ be an initialized irreducible homogeneous vector bundle with highest weight λ over $G/P(\alpha_k)$ of type B, C or D. Let $T_{k,\lambda} = (t_{ij})$ be its step matrix. Denote $n_l := \#\{t_{ij} | t_{ij} = l\}$. Then E_λ is an ACM bundle if and only if $n_l \geq 1$ for any integer $l \in [1, M_{k,\lambda}]$, where $M_{k,\lambda} = max\{t_{ij}\}$.

ACM bundle on isotropic Grassmannians

Example

Let E_μ be initialized homogeneous bundles with highest weight $\mu = 4\lambda_1 + 4\lambda_2$ on $OG(3,11) = B_5/P(\alpha_3)$. We can get that $T^B_{3,\mu} = (P^B_{3,\mu}, Q^B_{3,\mu}, R^B_{3,\mu})$, where

$$P^B_{3,\mu} = \begin{pmatrix} 1 & 2 \\ 6 & 7 \\ 11 & 12 \end{pmatrix}, Q^B_{3,\mu} = \begin{pmatrix} 3 & 4 \\ 8 & 9 \\ 13 & 14 \end{pmatrix}, R^B_{3,\mu} = \begin{pmatrix} \frac{5}{2} & 5 & \frac{15}{2} \\ 0 & \frac{15}{2} & 10 \\ 0 & 0 & \frac{25}{2} \end{pmatrix},$$

$M^B_{3,\mu} = 14$ and $n_l \geq 1$ for any integer $l \in [1, 14]$.

ACM bundle on isotropic Grassmannians

Corrolary

There are only finitely many irreducible homogeneous ACM bundles up to tensoring a line bundle over $G/P(\alpha_k)$ of types B, C and D. In particular, the moduli space of projective bundles produced by irreducible homogeneous ACM bundles consists of finite points.

Thank you!