Projective embedding of log pairs of Projective varieties and K-stability

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Shanghai, August 2022

Outline

1. Smooth case and Motivations
2. Logarithmic case
3. Gromov-Hausdorff limit

Conjecture (Yau-Tian-Donaldson)

Let (X, L) be a polarised manifold. $c_1(L)$ contains a constant scalar curvature Kähler (CSCK) metric if and only if (X, L) is K-polystable.

One direction

Theorem (Donaldson,Stoppa,Mabuchi)

If $c_1(L)$ contains a CSCK metric then (X, L) is K-polystable.

On the side of algebraic geometry

Chow Stability for subvarieties of $\mathbb{CP}^n$ which can be expressed by moment map

For any $z \in \mathbb{CP}^n$ with homogeneous coordinates $Z = [z_0, z_1, \cdots, z_n]^t$, the moment map for the $SU(n+1)$ action on $\mathbb{CP}^n$ is

$$\mu(z) = \frac{ZZ^*}{|Z|^2} - Trace \in Lie(SU(n+1))$$

Definition (Center of mass)

Let $V \subset \mathbb{CP}^n$ be a subvariety, then the center of mass of V is

$$\mu(V) = \int_V \frac{ZZ^*}{|Z|^2} d\mu_{FS} - Trace$$

V is called *balanced* if $\mu(V) = 0$

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Theorem (Chow-Stability)

V is Chow stable if and only if there is an $A \in SL(n+1;\mathbb{C})$ such that $A \cdot V$ is balanced.

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From differential geometry to algebraic geometry

The Bergman embedding:

- $\mathcal{H}_k$ the space of L_2-integrable holomorphic sections of L^k, with L_2-norm.
- $\{s_0, \cdots, s_N\}$ an orthonormal basis of $\mathcal{H}_k$
- the induced embedding $\Phi_k : X \to \mathbb{CP}^N$ by $\{s_0, \cdots, s_N\}$ is called the Bergman embedding.

Definition

A polarised manifold (X, L) is called balanced if some embedding $\Phi : X \to \mathbb{CP}^n$ given by a basis of $H^0(X, L)$ is balanced. And in that case $\Phi^*\omega_{FS}$ is called the balanced metric.

Theorem (Donaldson)

Let L be an ample line bundle over a projective complex manifold X with $\mathrm{Aut}(X, L)$ *discrete, then if* $\omega \in 2\pi c_1(L)$ *is a CSCK metric, then for* $k >> 1$, (X, L^k) *is balanced and the sequence of balanced metrics* ω_k *converges to* ω

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The other direction

- When $L = -K_X$ for a Fano manifold X, proved by Chen-Donaldson-Sun

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Donaldson's idea

Use Conical CSCK metric, (X, D, L, β), **where** D **is a divisor on** X, to do continuity method.

- Analytic part:**Conical CSCK metric**
- Algebraic part:**logarithmic K-Stabilities**

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Theorem (Odaka-Sun, Berman, Li-Sun)

When K_X is proportional to L, and $(K_X + D) \cdot L^{n-1} \geq 0$, then Then $(X, D, L, 0)$ is logarithmic K-semistable.

Theorem (Li-Wang)

Given a log Riemann surface (X, D) with $d \geq \chi(X)$, then for any ample line bundle L over X, $(X, D, L, 0)$ is logarithmic K-semistable.

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Theorem (S-Sun,J.Geom.Anal. 2021)

Given a log Riemann surface (X, D) with $d > \chi(X)$, then for any ample line bundle L over X, (X, D, L) is $\frac{2}{3}$-almost asymptotically Chow stable. More precisely, we have

$$\|\mu(\Phi_k(X), \Phi_k(D), \frac{2}{3})\|_2^2 = O(k^{-3/2}(\log k)^{121}).$$

where Φ_k is induced by an orthonormal basis of the Bergman space $\mathcal{H}_k$ of holomorphic sections of L^k that L^2 integrable with respect to the complete metric on $X \backslash D$ with negative constant curvature.
So $(X, D, L, 0)$ is logarithmic K-semistable.

Generalization to higher dimension

The case of projectivized line bundle: $(\hat{L}, D, A)$

- D, a smooth projective manifold.
- L, an ample line bundle over D.
- $\hat{L}$, the projective completion of L
- A, a polarization of $\hat{L}$ that admits a circle-invariant complete negative CSCK metric on the complement $\hat{L} \setminus D$, (constructed by Hwang-Singer using Calabi ansatz.)

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Theorem (S, Math. Ann. 2019)

$(\hat{L}, D, A, 0)$ *is K-semistable.*

Definition

For two compact metric spaces $(X, d_X), (Y, d_Y)$, the Gromov-Hausdorff distance of X and Y is defined as the infimum of the numbers ε such that there is a metric on $X \sqcup Y$ extending the metrics d_X and d_Y such that each of X and Y is ε-dense.

Definition

Let (X, d_X, p) and (X_i, d_{X_i}, p_i) be pointed metric spaces. We say (X_i, p_i) converges to (X, p) in the pointed Gromov-Hausdorff sense if

$$d_{GH}((\bar{B}_r^{X_i}, p_i), (\bar{B}_r^{X}, p)) \to 0 as\ i \to \infty$$

for all $r > 0$.

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$\mathcal{K}(n, c, V)$ consists of (X, g, J, L, A) satisfying the following conditions

- (X, g) a compact Riemannian manifold of real dimension $2n$, and volume of X being V
- J a complex structure with respect to which the metric is Kähler
- L a Hermitian line bundle over X, A is a connection on L with curvature $-i\omega$ where ω is the Kähler form. Satisfying $-\frac{1}{2}g \leq \text{Ric} \leq g$
- the "non-collapsing " condition:

$$\text{Vol}B_r \geq c\frac{\pi^n}{n!}r^{2n}$$

B_r is any r-ball in X.

Theorem (Donaldson-Sun)

Given n, c, V, there is a fixed k_1 and an integer N with the following effect:

- *Any X in $\mathcal{K}(n, c, V)$ can be embedded in a linear subspace of $\mathbb{CP}^N$ by sections of L^{k_1}.*
- *Let X_j be a sequence in $\mathcal{K}(n, c, V)$ with Gromov-Hausdorff limit X_∞. Then X_∞ is homeomorphic to a normal projective variety $W \subset \mathbb{CP}^N$. After passing to a subsequence and taking a suitable sequence of projective transformations, we can suppose that the projective varieties $X_j \subset \mathbb{CP}^N$ converge as algebraic varieties to W.*

a "collapsing" case

- (C_j, g_j) a sequence of compact genus $g \geq 2$ Riemann surfaces, with Riemannian metric g_j of constant Gaussian curvature -1;
- (C_0, g_0) a Punctured Riemann surface(not necessarily connected), g_0 a complete Riemannian metric of constant Gaussian curvature -1;
- (C_j, g_j) converges, in the topology of pointed Gromov-Hausdorff, to (C_0, g_0);
- As the Gaussian curvature is -1, the degeneration of metrics can only be "pinching a nontrivial loop", namely a sequence of surfaces with growingly thinner and longer handles, with the central loops degenerating to points.

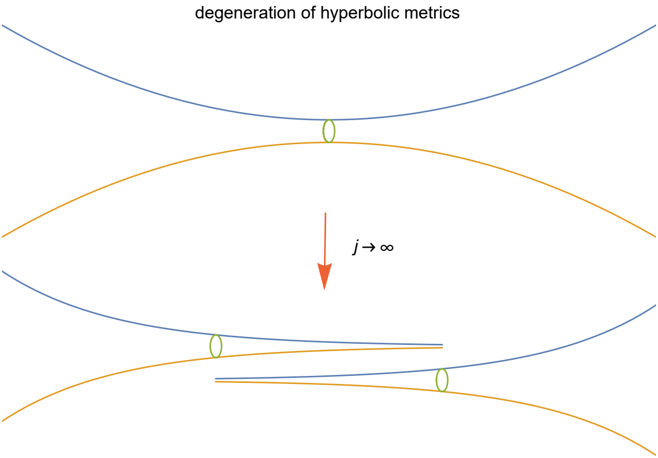

Figure: hyperbolic metric

- $\mathcal{H}_{j,k}$, space of L^2-integrable holomorphic sections of $K_{C_j}^k$
- For k large enough, a basis of $\mathcal{H}_{j,k}$ will induce a Kodaira embedding of C_j to $\mathbb{CP}^{N_k}$, where $N_k = \dim \mathcal{H}_{j,k} - 1$ is independent of $j \geq 1$
- For $j = 0$, the dimension of $\mathcal{H}_{j,k}$ is smaller than that of $j > 0$.
- So C_0 has d pairs of punctures, which will be called ends. And for k large enough,the dimension of $\mathcal{H}_{0,k}$ equals $N_k + 1 - d$.

Theorem (S, preprint)

For k large enough, we can choose an orthonormal basis for $\mathcal{H}_{j,k}$ for all $j > 0$, so that as $j \to \infty$ the image of the embedding

$$\Phi_{j,k} : C_j \to \mathbb{CP}^{N_k}$$

induced by the orthonormal basis converges to the image of C_0 under the embedding

$$\Phi_{0,k} : C_0 \to \mathbb{CP}^{N_k - d} \subset \mathbb{CP}^{N_k},$$

attached with d pairs of linear $\mathbb{CP}^1$'s. To each pair of the ends $(p_\alpha, p_{\alpha+d})$, a pair of linear $\mathbb{CP}^1$'s are associated, and form a connected chain connecting the images of these two points.

Remark

- *It is interesting to mention that during the process of taking limit, the pair of linear* $\mathbb{CP}^1$ *'s are developed as a pair of bubbles.*
- *This is illustrated by the following picture.*

degeneration of the embeddings

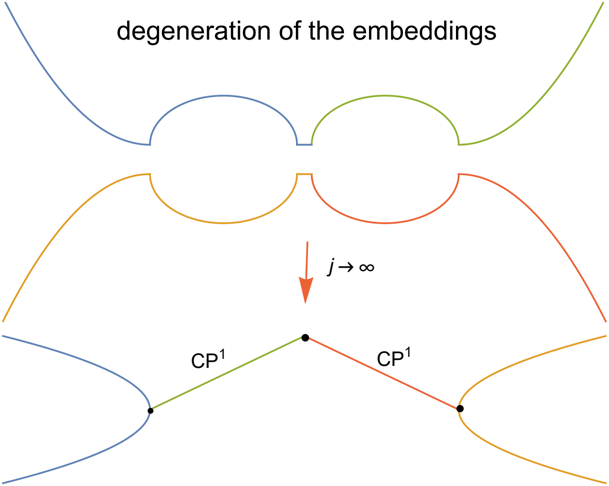

Figure: in $\mathbb{CP}^N$

Thank you

Thank you!

- $\mathcal{M}_g$ the moduli of smooth compact Riemann surfaces of genus $g \geq 2$.
- $\overline{\mathcal{M}_g}$ the Deligne-Mumford compactification of $\mathcal{M}_g$ consisting of stable curves.
- a stable curve is a compact connected Riemann surface whose only singularities are ordinary double points and whose automorphism group is finite.

In differential geometry

- Each smooth curve of genus g carries an unique Poincaré metric with constant Gaussian curvature -1.
- If C is a singular stable curve, then by removing the nodes, the smooth part carries an uniqu complete hyperbolic metric with constant Gaussian curvature -1.
- If a holomorphic family $\pi : \mathcal{C} \to D$ of compact smooth curves C_t degenerate to C_0, then with the hyperbolic metric, C_t converge to C_0 in the pointed Gromov-Hausdorff topology.