Li-Yau-Hamilton Estimates and Monotonicity

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Outline

1. LYH Estimates of trace type

In 1986, Li and Yau derived a gradient estimate for heat equation on Riemannian manifolds:

Theorem (Li-Yau, Acta Math.)

Let (M, g) be a complete Riemannian manifold with $Ric(M) \geq -K$ $(K \geq 0)$ and $u(x,t)$ be a positive solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \tag{1}$$

then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}. \tag{2}$$

If $K = 0$ and $\alpha \to 1$, then the estimate reduces to

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}. \tag{3}$$

Remark

It can be easily checked that if u is the fundamental solution on $\mathbb{R}^n$ given by the formula

$$u = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}},$$

then the equality holds. Moreover, it was shown by Ni that the estimate is sharp in the sense that the equality satisfies at some (x_0, t_0) implies that (M, g) must be isometric to $\mathbb{R}^n$.

Corollary (Li-Yau)

Let (M, g) be a complete Riemannian manifold with $Ric(M) \geq 0$ and $u(x, t)$ be a positive solution to the heat equation, then u satisfies

$$u(x_1, t_1) \leq \left(\frac{t_2}{t_1}\right)^{n/2} u(x_2, t_2) \exp\left(\frac{r^2(x_1, x_2)}{4(t_2 - t_1)}\right) \tag{4}$$

Theorem (Cao-Ljungberg-Liu, J. Funct. Anal.)

Let (M, g) *be a complete Riemannian manifold with* $Ric(M) \geq 0$ *and* $u(x,t)$ *be a positive solution to the nonlinear heat equation*

$$\frac{\partial}{\partial t} u = \Delta u + au \ln u. \tag{5}$$

Then in any of the three cases

- $a > 0$ *and* M *is closed*
- $a < 0$ *and* M *is closed*
- $a > 0$ *and* M *is complete noncompact*

the following estimate holds

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{an}{2(1 - e^{-at})}. \tag{6}$$

Theorem (Lu-Ni-Vázquez-Villani, JMPA)

Let (M, g) be a complete Riemannian manifold with $Ric(M) \geq 0$ and $u(x,t)$ be a positive solution to the porous medium equation

$$\frac{\partial}{\partial t} u = \Delta u^m, \quad m > 1. \tag{7}$$

Let

$$v = \frac{m}{m-1} u^{m-1}.$$

Then the following estimate holds

$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} \leq \frac{n(m-1)}{(n(m-1)+2)t}. \tag{8}$$

In terms of u, the estimate reduces to

$$m\frac{|\nabla u|^2}{u^{3-m}} - \frac{u_t}{u} \leq \frac{n}{(n(m-1)+2)t} \tag{9}$$

Theorem (Lu-Ni-Vázquez-Villani, JMPA)

Let (M, g) *be a complete Riemannian manifold with* $Ric(M) \geq 0$ *and* $u(x, t)$ *be a positive solution to the fast diffusion equation*

$$\frac{\partial}{\partial t} u = \Delta u^m, \quad 1 - \frac{2}{n} < m < 1. \tag{10}$$

Let

$$v = \frac{m}{m-1} u^{m-1}.$$

Then the following estimate holds

$$\frac{|\nabla v|^2}{v} - \frac{v_t}{v} \geq \frac{n(m-1)}{(n(m-1)+2)t}. \tag{11}$$

In terms of u*, the estimate reduces to*

$$m\frac{|\nabla u|^2}{u^{3-m}} - \frac{u_t}{u} \leq \frac{n}{(n(m-1)+2)t} \tag{12}$$

Theorem (Bailesteanu-Cao-Pulemotov, J. Funct. Anal.)

Suppose the manifold M is compact and $(M, g(x,t))$ is a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

Assume that $0 \leq Ric(x,t) \leq kg(x,t)$ for some $k > 0$. If u is a positive solution to heat equation then

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq kn + \frac{n}{2t}. \tag{13}$$

In 2011, Li and Xu proved a gradient estimate for heat equation on Riemannian manifolds with negative Ricci curvature lower bounded, which is sharper than that of Li-Yau.

Theorem (Li-Xu, Adv. Math.)

Let (M, g) be a complete Riemannian manifold with $Ric(M) \geq -K$ $(K > 0)$ and $u(x, t)$ be a positive solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \tag{14}$$

then u satisfies

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{\sinh Kt \cosh Kt - Kt}{\sinh^2 Kt}\right) \frac{u_t}{u} \leq \frac{nK}{2}(\coth Kt + 1). \tag{15}$$

2. Matrix LYH Estimates

Li-Yau's estimate can be rewritten as

$$\begin{aligned} & \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le \frac{n}{2t} \\ \Leftrightarrow \quad & \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \ge 0 \\ \Leftrightarrow \quad & g^{ij}\left(\frac{\nabla_i \nabla_j u}{u} - \frac{\nabla_i u \nabla_j u}{u^2} + \frac{1}{2t} g_{ij}\right) \ge 0 \\ \Leftrightarrow \quad & g^{ij}\left(\nabla_i \nabla_j \ln u + \frac{1}{2t} g_{ij}\right) \ge 0 \end{aligned}$$

Question: Does the positive solution u satisfy

$$\nabla_i \nabla_j \ln u + \frac{1}{2t} g_{ij} \ge 0? \tag{16}$$

In 1993, Hamilton extended Li-Yau's estimate to the full matrix version.

Theorem (Hamilton, Comm. Anal. Geom.)

Assume that (M, g) is a compact Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature. If u is a positive solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \tag{17}$$

then u satisfies

$$\nabla_i \nabla_j \ln u + \frac{1}{2t} g_{ij} > 0. \tag{18}$$

The assumption that (M, g) has parallel Ricci curvature, i.e. $\nabla_i R_{jk} = 0$, is quite restrictive, which essentially means that (M, g) is Einstein. When (M, g) is a Kähler manifold, this assumption can be dropped. In 2005, Cao and Ni extended the Li-Yau-Hamilton estimate to the Kähler manifolds.

Theorem (Cao-Ni, Math. Ann.)

Let (M, g) be a compact Kähler manifold with nonnegative holomorphic bisectional curvature. If u is a positive solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u,$$

then

$$\nabla_i \nabla_{\bar{j}} \ln u + \frac{1}{t} g_{i\bar{j}} \geq 0. \tag{19}$$

If the Kähler metric is evolved by the Kähler-Ricci flow, Chow and Ni proved the following estimate in 2007.

Theorem (Chow-Ni, J. Diff. Geom.)

Let $(M, g(t))$ be a compact solution to the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}$$

with nonnegative holomorphic bisectional curvature. If u is a positive solution to the forward conjugate heat equation

$$\frac{\partial}{\partial t} u = \Delta u + Ru,$$

then

$$\nabla_i \nabla_{\bar{j}} \ln u + R_{i\bar{j}} + \frac{1}{t} g_{i\bar{j}} > 0. \tag{20}$$

$$\frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} X^k + R_{i\bar{j},\bar{k}} X^{\bar{k}} + R_{i\bar{j}k\bar{l}} X^k X^{\bar{l}} + \frac{1}{t} R_{i\bar{j}} \geq 0. \tag{21}$$

Theorem of Cao-Ni can be connected with Theorem of Chow-Ni by following interpolation consideration.

Theorem (Chow-Ni, J. Diff. Geom.)

Let $(M, g(t))$ be a compact solution to the ε-Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -\varepsilon R_{i\bar{j}}$$

with nonnegative holomorphic bisectional curvature. If u is a positive solution to the parabolic equation

$$\frac{\partial}{\partial t} u = \Delta u + \varepsilon R u,$$

then

$$\nabla_i \nabla_{\bar{j}} \ln u + \varepsilon R_{i\bar{j}} + \frac{1}{t} g_{i\bar{j}} > 0. \tag{22}$$

To better understand how the Li-Yau-Hamilton estimate for the Ricci flow can be perturbed or extended, which is important in studying singularities of the Ricci flow, Chow and Hamilton extended matrix Li-Yau-Hamilton estimate on Riemannian manifolds to the constrained case in 1997.

Theorem (Chow-Hamilton, Invent. Math.)

Let (M, g) be a compact Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature. If u and v are solutions to the heat equation

$$\frac{\partial}{\partial t}u = \Delta u, \quad \frac{\partial}{\partial t}v = \Delta v,$$

with $|v| < u$, then

$$\nabla_i \nabla_j \ln u + \frac{1}{2t} g_{ij} > \frac{\nabla_i h \nabla_j h}{1 - h^2}, \quad h = v/u \tag{23}$$

It is natural to ask whether the Theorems of Cao-Ni and Chow-Ni can be generalized to constrained case. In 2015, we proved that

Theorem (R-Yao-Shen-Zhang, Math. Ann.)

Let $(M, g(t))$ be a compact solution of the ε-Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -\varepsilon R_{i\bar{j}} \tag{24}$$

with nonnegative holomorphic bisectional curvature. If u and v are solutions of the equation

$$\frac{\partial}{\partial t} u = \Delta u + \varepsilon R u, \qquad \frac{\partial}{\partial t} v = \Delta v + \varepsilon R v \tag{25}$$

with $|v| < u$, then we have

$$\nabla_i \nabla_{\bar{j}} \ln u + \frac{1}{t} g_{i\bar{j}} + \varepsilon R_{i\bar{j}} > \frac{\nabla_i h \nabla_{\bar{j}} h}{1 - h^2}, \tag{26}$$

where $h = v/u$.

In 2019, Wu generalized the matrix Li-Yau-Hamilton estimates to the nonlinear heat equation.

Theorem (Wu, Chin. Ann. Math.)

Let (M, g) be a compact Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature. If u is a positive solution to the equation

$$\frac{\partial}{\partial t} u = \Delta u + au \ln u, \tag{27}$$

then

$$\nabla_i \nabla_j \ln u + \frac{a}{2(1 - e^{-at})} g_{ij} > 0. \tag{28}$$

Then we considered the matrix Li-Yau-Hamilton estimates on Kähler manifolds.

Theorem (R, Arxiv:1911.00505)

Let (M, g) be a compact Kähler manifold with nonnegative holomorphic bisectional curvature. If u is a positive solution to the nonlinear heat equation

$$\frac{\partial}{\partial t}u = \Delta u + au\ln u, \tag{29}$$

then we have

$$\nabla_i\nabla_{\bar{j}}\ln u + \frac{a}{1-e^{-at}}g_{i\bar{j}} > 0. \tag{30}$$

Note that the Li-Yau-Hamilton estimate for the Kähler-Ricci flow is

$$\frac{\partial}{\partial t}R_{i\bar{j}} + R_{i\bar{k}}R_{k\bar{j}} + R_{i\bar{j},k}X^k + R_{i\bar{j},\bar{k}}X^{\bar{k}} + R_{i\bar{j}k\bar{l}}X^kX^{\bar{l}} + \frac{1}{t}R_{i\bar{j}} \geq 0. \tag{31}$$

So we use rescaled Kähler-Ricci flow estimate to couple the nonlinear heat equation.

Theorem (R, Arxiv:1911.00505)

Let $g(t)$ be a solution to the rescaled Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + a g_{i\bar{j}} \tag{32}$$

on compact manifold M with nonnegative holomorphic bisectional curvature. If u is a positive solution to the nonlinear heat equation

$$\frac{\partial}{\partial t} u = \Delta u + Ru + au \ln u, \tag{33}$$

where a is a positive constant, then we have

$$\nabla_i \nabla_{\bar{j}} \ln u + R_{i\bar{j}} + \frac{a}{1 - e^{-at}} g_{i\bar{j}} > 0. \tag{34}$$

$$\frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} X^k + R_{i\bar{j},\bar{k}} X^{\bar{k}} + R_{i\bar{j}k\bar{l}} X^k X^{\bar{l}} + \frac{ae^{-at}}{1 - e^{-at}} R_{i\bar{j}} \geq 0. \tag{35}$$

Theorem (R, Arxiv:1911.00505)

Let (M, g) *be a compact Kähler manifold. If* u *and* v *are positive solutions to the nonlinear heat equation*

$$\frac{\partial}{\partial t}u = \Delta u + au\ln u \tag{36}$$

and $h = v/u$*, then in any of the two cases:*

- $a > 0$, $0 < h < 1$ *and holomorphic bisectional curvature is nonnegative,*
- $a < 0$, $0 < c < h < 1$*, where* c *is a free parameter, and holomorphic bisectional curvature satisfies* $R_{i\bar{j}k\bar{l}} \geq -aK(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$ *with* K *satisfying* $K \geq -1 - \frac{2\ln c}{1-c^2}$*,*

the following estimate holds

$$\nabla_i\nabla_{\bar{j}}\ln u + \frac{a}{1-e^{-at}}g_{i\bar{j}} > \frac{\nabla_i h\nabla_{\bar{j}}h}{1-h^2}. \tag{37}$$

Theorem (R, Arxiv:1911.00505)

Let $(M, g(t))$ be a compact solution to the rescaled Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + a g_{i\bar{j}} \tag{38}$$

with nonnegative holomorphic bisectional curvature. If u and v are positive solutions to the nonlinear heat equation

$$\frac{\partial}{\partial t} u = \Delta u + Ru + au \ln u \tag{39}$$

with $v < u$ and $a > 0$, then we have

$$\nabla_i \nabla_{\bar{j}} \ln u + R_{i\bar{j}} + \frac{a}{1 - e^{-at}} g_{i\bar{j}} > \frac{\nabla_i h \nabla_{\bar{j}} h}{1 - h^2}, \tag{40}$$

where $h = v/u$.

These results can be extended to more general nonlinear heat equations on Riemannian manifolds.

Theorem (R, Arxiv:1911.00505)

Let (M, g) be a compact Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature and let L be a solution to the nonlinear heat equation

$$\frac{\partial}{\partial t}L = \Delta L + |\nabla L|^2 + F(L), \tag{41}$$

where F is a convex function of L, i.e., $F''(L) \geq 0$. If

$$\nabla_i \nabla_j L + f(t) g_{ij} \geq 0 \tag{42}$$

at $t = 0$, where $f(t)$ satisfies

$$2f^2(t) - F'(L)f(t) + f'(t) \geq 0, \tag{43}$$

then it persists for $t > 0$.

By taking $f(t) = \frac{1}{2t}$, then we have that

Corollary (R, Arxiv:1911.00505)

Let (M, g) be a compact Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature and let L be a solution to the nonlinear heat equation

$$\frac{\partial}{\partial t} L = \Delta L + |\nabla L|^2 + F(L), \tag{44}$$

where F is a function of L satisfying

$$F'(L) \leq 0, \quad F''(L) \geq 0.$$

Then we have that

$$\nabla_i \nabla_j L + \frac{1}{2t} g_{ij} \geq 0 \tag{45}$$

Example

Setting

$$F(L) = e^{kL}, \quad k \leq 0.$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} L &= \Delta L + |\nabla L|^2 + e^{kL} \\ &\Downarrow L = \ln u \\ \frac{\partial}{\partial t} u &= \Delta u + u^{k+1}, \ \Rightarrow \ \nabla_i \nabla_j \ln u + \frac{1}{2t} g_{ij} \geq 0 \end{aligned} \tag{46}$$

- The term
$$2f^2(t) - F'(L)f(t) + f'(t)$$
appears as the coefficient of g_{ij}. In order to get rid of it, we need the condition that
$$2f^2(t) - F'(L)f(t) + f'(t) \geq 0.$$
- If $F = 0$, then we have $f(t) = \frac{1}{2t}$ by solving
$$2f^2(t) + f'(t) = 0.$$
- If $F = aL$, then we have $f(t) = \frac{a}{2(1-e^{-at})}$ by solving
$$2f^2(t) - af(t) + f'(t) = 0.$$

The proof is standard with a little modification.

$$\begin{aligned} &\nabla_i \nabla_j \ln u + f(t) g_{ij} \geq 0 \\ \Longleftrightarrow &\nabla_i \nabla_j u - \frac{\nabla_i u \nabla_j u}{u} + f(t) u g_{ij} \geq 0 \end{aligned} \tag{47}$$

Instead of considering the evolution equation of

$$\nabla_i \nabla_j u - \frac{\nabla_i u \nabla_j u}{u} + f(t) u g_{ij},$$

we consider the evolution equation of

$$\nabla_i \nabla_j \ln u + f(t) g_{ij}.$$

Similar technique can be applied to consider the Li-Yau-Hamilton estimate of trace type.

3. Monotonicity

- The frequency functional for harmonic functions in $\mathbb{R}^n$ was introduced by Almgren. It is defined by

$$I(r) = \frac{r \int_{B(x,r)} |\nabla u|^2 d\mu}{\int_{\partial B(x,r)} |u|^2 dA}, \tag{48}$$

  Almgren showed that $I(r)$ is monotone increasing in r, which measures the rate of growth of harmonic functions.

- The parabolic frequency functional for solutions of heat equation and other parabolic equation was introduced by Poon.

$$I(R) = \frac{R^2 \int_{t=t_0-R^2} |\nabla u|^2 G_{x_0,t_0}(x,t)dx}{\int_{t=t_0-R^2} u^2 G_{x_0,t_0}(x,t)dx}, \tag{49}$$

  where

$$G_{x_0,t_0}(x,t) = (t_0 - t)^{-n/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

  is the backward heat kernel on $\mathbb{R}^n$.

Theorem (Poon,Ni)

Assume that M is a Riemmannian manifold with nonnegative sectional curvature and parallel Ricci curvature. Let u be a solution to the heat equation. For any (x_0, t_0), let $\tau = t_0 - t$. Let $H(x, \tau; x_0, 0)$ be the fundamental solution to the backward heat equation. Define

$$Z_2(t) = \int_M H(x, \tau; x_0, 0) u^2(x, t) d\mu(x),$$

$$D_2(t) = \int_M H(x, \tau; x_0, 0) |\nabla_x u(x, t)|^2 d\mu(x),$$

$$I_2(t) = \frac{\tau D_2(t)}{Z_2(t)}.$$

Then $\frac{d}{dt} I_2 \leq 0$.

Theorem (Ni)

Assume that M is a Riemmannian manifold with nonnegative sectional curvature and parallel Ricci curvature. Let u be a harmonic function of polynomial growth. Define

$$Z_2(x,t) = \int_M H(x,y,t)u^2(y)d\mu(y),$$

$$D_2(x,t) = \int_M H(x,y,t)|\nabla_y u|^2(y)d\mu(y),$$

$$I_2(x,t) = \frac{tD_2(x,t)}{Z_2(x,t)},$$

where $H(x,y,t)$ is the fundamental solution to the heat equation. Then $\frac{d}{dt}I_2 \geq 0$, which implies that $\ln Z_2$ is an increasing function and convex in $\ln t$.

Recently Colding and Minicozzi observed that

$$
\begin{aligned}
Z_2(R) &= \int u^2 G_{x_0,t_0}(x,t)dx \\
&= R^{-n}\int u^2(y,-R^2)e^{-\frac{|y|^2}{4R^2}}dy \\
&= \int u^2(e^{-\frac{s}{2}}x,-e^{-s})e^{-\frac{|x|^2}{4}}dx \\
&= \int w^2(x,s)e^{-\frac{|x|^2}{4}}dx = Z_2(s)
\end{aligned} \tag{50}
$$

by letting

$$
w(x,s) = u(e^{-\frac{s}{2}}x,-e^{-s}), y = e^{-\frac{s}{2}}x, R = e^{-\frac{s}{2}}
$$

Colding and Minicozzi considered drift Laplacian on vector-valued functions $u: M \to \mathbb{R}^N$ by

$$\mathcal{L}_\phi u = \Delta u - <\nabla u, \nabla \phi> = e^\phi div(e^{-\phi} \nabla u) \tag{51}$$

on a Riemannian manifold (M, g), where $\phi: M \to \mathbb{R}$ is a smooth function. Define

$$Z_2(t) = \int |u|^2 e^{-\phi}, \tag{52}$$

$$D_2(t) = -\int |\nabla u|^2 e^{-\phi}, \tag{53}$$

$$I_2(t) = \frac{D_2(t)}{Z_2(t)}. \tag{54}$$

Then Colding and Minicozzi proved that

Theorem (Colding-Minicozzi, IMRN)

If u is a smooth solution to the drift heat equation

$$\frac{\partial}{\partial t}u = \mathcal{L}_\phi u, \tag{55}$$

then $\frac{d}{dt}(\log Z_2(t)) = 2I_2(t)$ and $\frac{d}{dt}I_2(t) \geq 0$, which implies that $\log Z_2(t)$ is convex. Moreover, if $I_2(t) = \lambda$ is a constant, then $u(x,t) = e^{\lambda t}u(x,0)$ and $u(x,0)$ is an eigenfunction of $\mathcal{L}_\phi$ with eigenvalue $-\lambda$.

Furthermore, they extended the theorem for dift Laplacian to more general operators satisfying

$$|(\frac{\partial}{\partial t} - \mathcal{L}_\phi)u| \le C(t)(|u| + |\nabla u|). \tag{56}$$

Theorem (Colding-Minicozzi, IMRN)

If $u : M \times [a, b] \to \mathbb{R}^N$ is a smooth function satisfying (56), then

$$\frac{d}{dt}(\log Z_2(t)) \ge (2 + \frac{C}{2})I_2 - \frac{3C}{2},$$
$$\frac{d}{dt} I_2(t) \ge C^2(I_2 - 1),$$
$$C^2 \ge \frac{d}{dt}(\log(1 - I_2)).$$

The weighted p-Laplacian for $p \geq 2$ is defined by

$$\Delta_{p,\phi} u = e^{\phi} div(|\nabla u|^{p-2} e^{-\phi} \nabla u). \tag{57}$$

It is obvious that if $p = 2$, then weighted p-Laplacian reduces to drift Laplacian. Define the following quantities:

$$\begin{aligned} Z_p(t) &= \int |u|^2 e^{-\phi}, \\ D_p(t) &= \int |\nabla u|^p e^{-\phi}, \\ I_p(t) &= \frac{D_p(t)}{Z_p(t)}. \end{aligned} \tag{58}$$

Theorem (Liu-Xu)

If $u : M \times [a, b] \to \mathbb{R}^N$ satisfies

$$\partial_t u = \Delta_{p,\phi} u, \tag{59}$$

then $\frac{d}{dt}(\log Z_p(t)) = 2I_p(t)$ and $\frac{d}{dt} I_p(t) \geq 0$ so $\log Z_p(t)$ is convex. Moreover, if $I_p(t) = \lambda$ is a constant, then $u(x, t) = e^{\lambda t} u(x, 0)$ and $u(x, 0)$ is an eigenfunction of $\Delta_{p,\phi}$ with eigenvalue $-\lambda$.

Moreover, we extended the theorem for weighted p-Laplacian to more general operators satisfying

$$|(\partial_t - \Delta_{p,\phi})u| \leq C(t)(|u| + |\nabla u|^{\frac{p}{2}}), \tag{60}$$

where $C(t)$ is allowed to depend on t.

Theorem (Liu-Xu)

If $u : M \times [a, b] \to \mathbb{R}^N$ satisfies (60), then

$$\begin{aligned}
&\frac{d}{dt}(\log Z_p(t)) \geq (2 + \frac{C}{2})I_p - \frac{3C}{2}, \\
&\frac{d}{dt}I_p(t) \geq C^2(I_p - 1), \\
&C^2 \geq \frac{d}{dt}(\log(1 - I_p)).
\end{aligned}$$

Theorem (Ni)

Let M is a complete Kähler manifold with nonnegative bisectional curvature. Let f be a holomorphic function of finite order and $H(x, y, t)$ be the heat kernel. For any $p > 0$, define

$$Z_p(x,t) = \int_M H(x,y,t)|f|^p(y)d\mu(y),$$

$$D_p(x,t) = \frac{p}{4}\int_M H(x,y,t)|\nabla f|^2|f|^{p-2}(y)d\mu(y),$$

$$I_p(x,t) = \frac{tD_p(x,t)}{Z_p(x,t)},$$

Then $\frac{d}{dt} I_p \geq 0$, which implies that $\frac{1}{p}\ln Z_p(x,t)$ is an increasing, convex function of $\ln t$.

Let $(M, g(t))$ be a complete solution to the ε-Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -\varepsilon R_{i\bar{j}} \tag{61}$$

and $H(x, t; x_0, 0)$ be the fundamental solution of parablic equation

$$\frac{\partial}{\partial t} u = \Delta u + \varepsilon R u. \tag{62}$$

If f is a holomorphic function of finite order, define the following quantities:

$$\begin{aligned} Z_p(t) &= \int_M H(x, t; x_0, 0)|f|^p(x) d\mu, \\ D_p(t) &= \frac{p}{4} \int_M H(x, t; x_0, 0)|\nabla f|^2 |f|^{p-2}(x) d\mu, \\ I_p(t) &= \frac{t D_p(x, t)}{Z_p(t)}. \end{aligned} \tag{63}$$

Theorem (Liu-Xu)

Let $(M, g(t))$ be a complete solution to the ε-Kähler-Ricci flow on a Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Then for any $p > 0$, $\frac{1}{p}\frac{\partial}{\partial t}(\log I_p(x,t)) = \frac{D_p(t)}{I_p(t)}$ is an increasing, convex function of $\log t$.

Corollary

Let $(M, g(t))$ be a complete solution to ε-Kähler-Ricci flow on a Kähler manifold with nonnegative and bounded bisectional curvature. If f is a holomorphic function of finite order, then

$$\log I_0(t) = \int_M H(x,t;x_0,0)\log|f|(x)d\mu(x) \tag{64}$$

is an increasing, convex function of $\log t$.

Based on the matrix Li-Yau-Hamilton estimate for the Ricci flow on surface

$$\nabla_i \nabla_i \ln R + \frac{1}{2}(R + \frac{1}{t})g_{ij} > 0,$$

Li and Wang proved that

Theorem (Li-Wang, CVPDE)

Let M be a closed surface. Suppose that $g(t)$ is a solution to the Ricci flow $\frac{\partial}{\partial t} g = -Rg$ with positive scalar curvature. Let u be a nontrivial solution to the backward heat equation $\frac{\partial}{\partial t} u + \Delta u = 0$. Define

$$Z_2(t) = \int_M R(x,t) u^2(x,t) d\mu(x),$$
$$D_2(t) = \int_M R(x,t) |\nabla_x u(x,t)|^2 d\mu(x),$$
$$I_2(t) = \frac{t D_2(t)}{Z_2(t)}.$$

Then $\frac{d}{dt} I_2 \geq 0$.

Since there is a matrix Li-Yau-Hamilton estimate for the Kähler-Ricci flow

$$\nabla_i \nabla_{\bar{j}} R + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} \nabla_{\bar{k}} \ln R + R_{i\bar{j},\bar{k}} \nabla_k \ln R + R_{i\bar{j}k\bar{l}} \nabla_{\bar{k}} \ln R \nabla_l \ln R + \frac{1}{t} R_{i\bar{j}} \geq 0, \tag{65}$$

we propose following open problem

Open Problem

Let M be a complete noncompact Kähler manifold. Suppose that $g(t)$ is a solution to the Kähler-Ricci flow $\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}$ with positive holomorphic bisectional curvature. Define

$$Z_p(t) = \int_M R(x,t)|f|^p(x)d\mu(x),$$
$$D_p(t) = \int_M R(x,t)|\nabla f|^2|f|^{p-2}(x)d\mu(x),$$
$$I_p(t) = \frac{tD_p(t)}{Z_p(t)}.$$

Does there hold $\frac{d}{dt} I_2 \geq 0$?

THANK YOU !