Geometric quantization on complex manifolds with boundary

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Plan of the talk:

1. Quantization commutes with reduction (Guillemin-Sternberg)
2. Geometric quantization on CR manifolds (Hsiao-Ma-Marinescu)
3. Geometric quantization on complex manifolds with boundary (Hsiao-Huang-Li-S.)

1. Quantization commutes with reduction (Guillemin-Sternberg)

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- To quantize a compact symplectic manifold, i.e., to associate a Hilbert space $\mathcal{H}$, we need a (prequantum) line bundle whose first Chern form equals to the symplectic form.
- (L, h^L, ∇^L) a complex line bundle over M equipped with a Hermitian metric h^L and a Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega, \qquad \text{where} \quad R^L := \nabla^L \circ \nabla^L.$$

L is called the prequantum line bundle on (M, ω).

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- Let (M, ω, J) be Kähler, J the complex structure map of M, ω and J are compatible,

  i.e. $\omega(\cdot, J\cdot)$ is positive definite and L is holomorphic.
- The quantization of (M, ω, J) is defined as

$$H^0(M, L) := \left\{u \in C^\infty(M, L); \overline{\partial}u = 0\right\},$$

  the space of holomorphic sections of the prequantum line bundle L.

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- The orthogonal projection on $H^0(M, L^m)$ palys a crucial role in the correspondence.

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- For $\xi \in \mathfrak{g}$, let ξ_M be the vector field on M generated by ξ.
- $\mu : M \to \mathfrak{g}^*$ the moment map induced by ω is defined by

$$\iota_{\xi_M}\omega = d\langle \mu, \xi \rangle \qquad \text{for all} \quad \xi \in \mathfrak{g}.$$

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- Guillemin-Sternberg conjecture:

$$\begin{array}{ccc} (M, L) & \xrightarrow{\text{quantization}} & H^0(M, L^m) \\ \big\downarrow \text{reduction} & & \big\downarrow \text{reduction} \\ (M_G, L_G) & \xrightarrow{\text{quantization}} & H^0(M_G, L_G^m) \sim H^0(M, L^m)^G \end{array}$$

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Theorem ([Q, R]=0, Guillemin-Sternberg (1982))

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- For general G, Meinrenken, Meinrenken-Sjamaar (1998), symplectic cut techniques of Lerman.

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- For R^L non-degenerate and m large (more on CR manifold), Hsiao-Huang (2021).

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- $H^0_b(X)^G$: the space of the G-invariant L^2 CR functions.

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- For every $m \in \mathbb{Z}$, let
$$H^0_{b,m}(X)^G := \left\{u \in H^0_b(X)^G;\ (e^{i\theta})^* u = e^{im\theta} u, \text{for every } e^{i\theta} \in S^1\right\}.$$
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- Grauert: For every $m \in \mathbb{Z}$,
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- From Guillemin-Sternberg theorem,
$$H^0_{b,m}(X)^G \cong H^0_{b,m}(X_G), \quad \text{for every } m \in \mathbb{Z}.$$
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- But the quantum spaces consisting of CR functions and are infinite dimensional.

- Hsiao, Ma and Marinescu (2019) use Szegő kernel method and G-invariant microlocal Fourier integral operator calculation to tackle the problem.

2. Geometric quantization on CR manifolds (Hsiao-Ma-Marinescu)

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Definition

We say that $T^{1,0}X$ is a CR structure of X if

(i) $\dim_{\mathbb{C}} T_x^{1,0}X = n$, for every $x \in X$.

(ii) $T^{1,0}X \bigcap T^{0,1}X = \{0\}$, where $T^{0,1}X := \overline{T^{1,0}X}$.

(iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, $\mathcal{V} = \mathcal{C}^\infty(X, T^{1,0}X)$.

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If we can find a CR structure $T^{1,0}X$ on X, we call the pair $(X, T^{1,0}X)$ a CR manifold.

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  - $\mathbb{C}T^*X = T^{*1,0}X \oplus T^{*0,1}X \oplus \mathbb{C}\omega_0$, $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$,

    where $T^{*0,1}X$: bundle of $(0,1)$ forms and ω_0: Reeb one form.

$$\langle \omega_0 \, , \, T \rangle = -1, \qquad T^{*0,1}X = (T^{1,0}X \oplus \mathbb{C}T)^{\perp}.$$

Levi form

Definition

For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T_p^{1,0}X$ given by

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- We extend $\overline{\partial}_b$ to L^2 space:
$$\overline{\partial}_b : \mathrm{Dom}\,\overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X),$$
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  where $\mathrm{Dom}\,\overline{\partial}_b = \{u \in L^2(X); \overline{\partial}_b u \in L^2(X)\}$.
- For $u \in L^2(X)$, we say that u is a CR function if $u \in \mathrm{Ker}\,\overline{\partial}_b$.
- If X is strongly pseudoconvex at some point of X and $\overline{\partial}_b$ has L^2 closed range, then $\dim \mathrm{Ker}\,\overline{\partial}_b = +\infty$ (Boutet de Monvel-Sjöstrand, Hsiao-Marinescu).

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- Goal: Study $H_b^0(X)^G$: the space of global G-invariant L^2 CR functions.

CR momentum map

Definition

The momentum map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

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Assumption

0 *is a regular value of* μ, G *acts on* $\mu^{-1}(0)$ *freely and the Levi form of* X *is positive definite near* $\mu^{-1}(0)$.

- X is not necessarily strongly pseudoconvex.

Example

- Let $X = \{(z_1, z_2, z_3) \in \mathbb{C}^3;\ |z_1|^4 + |z_2|^2 + |z_3|^2 = 1\}$.

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- X admits a S^1-action:
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Suppose that R^L is positive near $\mu^{-1}(0)$. Then, for m large,

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Theorem (Hsiao-Huang)

Assume that R^L has q negative eigenvalues and $n-q$ positive eigenvalues and $R^L|_{M_G}$ has $q-r$ negative eigenvalues and $n-d-q+r$ positive eigenvalues. Then, for m large,

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3. Geometric quantization on complex manifolds with boundary (Hsiao-Huang-Li-S.)

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where $T^{1,0}M'$ denotes the holomorphic tangent bundle of M'.

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- Take a Hermitian metric $\langle \cdot | \cdot \rangle$ for M'. Then $\langle \cdot | \cdot \rangle$ induces by duality a Hermitian metrics on $\mathbb{C}T^*M'$ and on $\wedge^q T^{*0,1}M'$. We denote all these Hermitian metrics by $\langle \cdot | \cdot \rangle$.

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- Let $\rho \in C^\infty(M', \mathbb{R})$ be a defining function of X, that is,

$$X = \{x \in M';\ \rho(x) = 0\}, \quad M = \{x \in M';\ \rho(x) < 0\}$$

and $d\rho(x) \neq 0$ at every point $x \in X$ such that

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- Thus, the Levi-form on X is exactly
$$\mathcal{L}_p(U, V) = \langle \partial\overline{\partial}\rho(p)\, ,\, U \wedge \overline{V} \rangle, \quad U, V \in T_p^{1,0}X.$$

Holomorphic functions smooth up to the boundary

- The Cauchy-Riemann operator

$$\overline{\partial} : \mathcal{C}^{\infty}(\overline{M}) \to \Omega^{0,1}(\overline{M}),$$

where $\mathcal{C}^{\infty}(\overline{M}) := \mathcal{C}^{\infty}(M')|_M$ and $\Omega^{0,1}(\overline{M}) := \Omega^{0,1}(M')|_M$.

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Definition

The momentum map associated to the form ω_0 is the map $\mu : M' \to \mathfrak{g}^*$ such that, for all $x \in M'$ and $\xi \in \mathfrak{g}$, we have

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0 is a regular value of μ, G acts on $\mu^{-1}(0)$ freely and $i\partial\overline{\partial}\rho$ is positive definite near $\mu^{-1}(0)$.

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- Develop some kind of G-invariant microlocal F.I.O. method.

G-invariant Bergman kernel asymptotics

- The following Theorem generalizes classical work of Fefferman to G-invaraint case.

Theorem III (Hsiao-Huang-Li-S.)

- *B_G is smoothing outside $\mu^{-1}(0) \cap X$.*
- *In an open set U of $\mu^{-1}(0) \cap X$, we have*

$$B_G(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t} a(x,y,t)dt \quad \text{on } U \times U,$$

  - $a(x,y,t) \sim \sum_{j=0}^\infty a_j(x,y) t^{n-\frac{d}{2}-j} \quad \text{in } S_{1,0}^{n-\frac{d}{2}}(U \times U \times \mathbb{R}_+),$ $a_0(x,y) \neq 0$
  - $d_x\Phi(x,x) = -d_y\Phi(x,x) = -\omega_0(x), \qquad \forall x \in \mu^{-1}(0) \cap X,$
  - $\operatorname{Im}\Phi(x,y) \geq 0, \qquad \operatorname{Im}\Phi(x,x) \approx d(x, \mu^{-1}(0) \cap X)^2.$

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Thanks a lot!