Lecture of Yum-Tong Siu of Harvard University
at the Several Complex Variables Conference
in Shanghai University
on August 20, 2022
from 8:30 a.m. to 9:20 a.m. Shanghai Time

TITLE: Some Problems in Several Complex Variables

ABSTRACT: I will discuss three directions of research in several complex variables which I have been working on.

(1) We discuss the use of the gradient term in the Bochner-Kodaira formula with boundary term. Two potentially important applications are the strong rigidity for holomorphic vector bundles and the very ampleness part of the Fujita conjecture. As an illustration we use the gradient term to construct holomorphic sections to prove the Thullen-type extension across codimension 1 for holomorphic vector bundles with Hermitian metric whose curvature is L^p for some $p > 1$.

(2) The global non-deformability of irreducible compact Hermitian symmetric manifolds has been a longstanding open conjecture, known for the complex projective space and the hyperquadric or for Kähler deformation. To handle the Grassmannians and the other open cases, we discuss a new technique of using jets of high order, iterated Euler vector fields, and eigenvalues with respect to background frames of reference.

(3) We discuss the use of the stabilization of irreducible components of Lelong sets in problems of finite generation and the abundance conjecture.

The usual format of a talk in a conference is to state the speaker's result (which is normally already published or posted on the ArXiv), give some background information to highlight what is new, and sketch the main points of the proof when time permits.

Many times people have asked me what I think is the future and the directions of research in several complex variable. Of course, any answer to such a question is only from the personal perspective of the answerer and is completely surjective. I chose the format of this talk, with the answer to such a question in mind. Within the time slot available, I will talk about three directions which I have been looking at recently. I will focus on the main ideas and techniques and leave the precise statements to references which I will indicate where to find in the public domain.

I hope that your time spent in listening to my talk is worth your while and you gain some useful information, otherwise not available. A better hope is that you are incentivized by the talk to consider going into one of the directions of research.

Part I. *Gradient Term in Bochner-Kodaira Formula.*

Bochner-Kodaira formula with Boundary Term. Solving the $\bar\partial$ equation is one of the most fundamental techniques in several complex variables. Kodaira's vanishing and embedding theorems depend on it. For a complex manifold X with a holomorphic line bundle L with a Hermitian metric and a smooth L^2 $(L+K_X)$-valued $(0,1)$-form g on a relatively compact domain Ω in X which belongs to the domain of the actual adjoint of $\bar\partial$, the Bochner-Kodaira formula with boundary term is

$$\left\|\bar\partial g\right\|_\Omega^2+\left\|\bar\partial^* g\right\|_\Omega^2=(\mathrm{Levi}_{\partial\Omega} g,\, g)_{\partial\Omega}+\left\|\bar\nabla g\right\|_\Omega^2+(\Theta_L\, g,\, g)_\Omega\,,$$

where $\mathrm{Levi}_{\partial\Omega}$ is the Levi form of the boundary $\partial\Omega$ and Θ_L is the curvature of L.

Usually only the curvature term on the right-hand side of this formula is used to get vanishing theorems. The boundary term was used by Kohn in his pioneering work on the differential relations for multiplier sheaves for weakly pseudoconvex domains. His differential relations are from the use of the Levi form on the boundary.

Joseph J. Kohn, Subellipticity of the $\bar\partial$-Neumann problem on pseudo-convex domains: sufficient conditions. *Acta Math.* **142** (1979), 79 – 122.

Kohn's differential relations can be interpreted in terms of an osculated version of Noether's theorem on conservation laws, but we will not talk about it at all today.

An important point is the if g is an L-valued $(0,1)$-form instead of $(L+K_X)$-valued $(0,1)$-form, the formula becomes

$$\left\|\bar\partial g\right\|_\Omega^2+\left\|\bar\partial^* g\right\|_\Omega^2=(\mathrm{Levi}_{\partial\Omega} g,\, g)_{\partial\Omega}+\left\|\bar\nabla g\right\|_\Omega^2+(\Theta_L\, g,\, g)_\Omega+(\mathrm{Ricci}_X\, g,\, g)_\Omega\,,$$

where the Ricci curvature as curvature of $-K_X$ enters when L is replaced by $L - K_X$ in the formula for $(L + K_X)$-valued $(0,1)$-form.

I would like to point out that the techniques of $\bar\partial$ estimates is a generalization of Cramer's rule with compatibility condition.

Cramer's Rule with Compatibility Condition. Consider $Tx = f$ subject to the compatibility condition $Sf = 0$ in the case of linear operators in finite-dimensional vector spaces (in the sense that $Tx = f$ is solvable for x precisely when $Sf = 0$). The generalized Cramer's rule gives

$$x = T^*(TT^* + S^*S)^{-1}f,$$

which is a simple standard argument in basic linear algebra, though almost never mentioned in a high-school or beginning college course. In the case of $S = 0$, it becomes the usual Cramer's rule

$$x = T^*(TT^*)^{-1}f = T^{-1}f.$$

The inequality

$$\|T^*g\|^2 + \|Sg\|^2 \geq c\|g\|^2$$

for $\bar\partial$-estimates is for the inversion of $TT^* + S^*S$. The operator T^* occurs because $\bar\partial$ equation is solved in the sense of a weak solution.

Nadel's Vanishing Theorem. Instead of a strictly positive lower bound to invert $TT^* + S^*S$, a good weight function can be used to yield the following. Let X be a compact complex algebraic manifold and L be a holomorphic line bundle with (possibly singular) Hermitian metric $e^{-\varphi}$ whose curvature current dominates a smooth strictly positive $(1,1)$-form on X. Then $H^p(X, \mathcal{I}_\varphi(L + K_X)) = 0$ for $p \geq 1$, where $\mathcal{I}_\varphi$ is the multiplier ideal sheaf consisting all of holomorphic function germs f with $|f|^2e^{-\varphi}$ locally integrable.

Alan Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. *Ann. of Math.* **132** (1990), 549 – 596.

The multipliers for Nadel's vanishing are different from Kohn's multipliers. Nadel's vanishing can be obtained by removing a tubular neighborhood of some ample complex hypersurface, which is a standard procedure in analytic methods in several complex variables.

Two Potentially Important Applications of the Use of the Gradient term. One of the main statements in the strong rigidity theory for compact Kähler manifolds is that if X is a compact Kähler manifold of complex dimension ≥ 2 whose curvature satisfies some appropriate curvature condition, then any compact Kähler manifold Y which is of the same homotopy type as X must be biholomorphic or conjugate biholomorphic to X.

Yum-Tong Siu, The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. of Math.* **112** (1980), 73 – 111.

The key argument is to apply a form of the Bochner-Kodaira formula to a harmonic map from Y to X which is a homotopy equivalence.

A natural question is to ask for an analogous strong rigidity theory for holomorphic vector bundles E over a fixed compact Kähler (or even algebraic) manifold M.

One seeks an appropriate condition on E so that a holomorphic vector bundle F over M which is topologically equivalent to E should be biholomorphic or conjugate holomorphic to E. The obstacle is that a condition for the infinitesimal rigidity of E is the vanishing of $H^1(M, E \otimes E^*)$ which cannot be obtained as a consequence of the curvature term in the Bochner-Kodaira formula, because the curvature is for the tensor product of E and its dual E^*. For such an approach, the strong rigidity theory or even the infinitesimal rigidity for E must be from the use of the gradient term of the Bochner-Kodaira formula. How it is to be done remains an open problem. It is a difficult problem and we are unable to discuss it here.

The second potentially important application of the use the gradient term is the very ampleness part of the Fujita conjecture, which is easier but still quite a challenge, interpretable as *Sobolev estimates in algebraic geometry.* In my view it is a good direction to go into.

For a compact algebraic manifold X of complex dimension n and an ample line bundle L over X, the original Fujita conjecture seeks to prove that $mL + K_X$ is globally free for $m \geq n+1$ and is very ample for $m \geq n+2$. The freeness part is known under the stronger assumption of $m \geq \frac{1}{2}n(n+1)+1$ (which has been weakened a bit over the years)

Urban Angehrn and Yum-Tong Siu, Effective freeness and point separation for adjoint bundles. *Invent. Math.* **122** (1995), 291 – 308.

To generate a higher order jets at prescribed points P_j by $\Gamma(X, L+K_X)$, one needs to get the vanishing of $H^1(X, \mathcal{I}_\varphi(L+K_X))$ for a metric $e^{-\varphi}$ for L with nonnegative curvature current such that $\mathcal{I}_\varphi$ behaves near P_j like certain powers of the maximum ideals $\mathfrak{m}_{P_j}^{k_j}$ at P_j. When L is ample, the Riemann-Roch and Kodaira's vanishing yield such a metric for $mL+K_X$ for m effectively sufficiently large and, by taking roots, also for $m_\varepsilon L+\varepsilon K_X$ for any $\varepsilon > 0$. To use only the curvature term of the Bochner-Kodaira formula, higher order jets at prescribed points can only be generated by $\Gamma(X, (m_\varepsilon L+\varepsilon K_X)+K_X)$ and not by $\Gamma(X, m_\varepsilon L+K_X)$, unless for the metric of $m_\varepsilon L = (m_\varepsilon L+\varepsilon K_X) - \varepsilon K_X$ one uses the above metric for $m_\varepsilon L+\varepsilon K_X$ and the metric for $-\varepsilon K_X$ whose curvature is εRicci so that

$$\left\|\bar\partial g\right\|_\Omega^2+\left\|\bar\partial^* g\right\|_\Omega^2 = (\mathrm{Levi}_{\partial\Omega} g,\, g)_{\partial\Omega}+\left\|\bar\nabla g\right\|_\Omega^2+(\Theta_{m_\varepsilon L+\varepsilon K_X}\, g,\, g)_\Omega+\varepsilon\,(\mathrm{Ricci}_X\, g,\, g)_\Omega$$

for $m_\varepsilon L+K_X$-valued $(0,1)$-form g. The key step is to use the gradient term $\left\|\bar\nabla g\right\|_\Omega^2$ to dominate $\varepsilon\,(\mathrm{Ricci}_X\, g,\, g)_\Omega$ for $\varepsilon>0$ effectively sufficiently small.

Use of Gradient Term Depends on Cauchy Integral Formula for Smooth Functions. Let D be a smooth bounded domain in $\mathbb{C}$ with coordinate z_1. Cauchy's integral formula for (nonholomorphic) smooth functions applied to a (nonholomorphic) function $\Phi(z_1)$ on $\bar D$ is

$$\Phi(z_1) = \frac{1}{2\pi i}\int_{\zeta_1\in\partial D}\frac{\Phi(\zeta_1)d\zeta_1}{\zeta_1-z_1}+\frac{1}{2\pi i}\int_{\zeta_1\in D}\frac{(\partial_{\bar\zeta_1}\Phi(\zeta_1))d\zeta_1\wedge d\bar\zeta_1}{\zeta_1-z_1}$$

for $z_1\in D$. The condition for a $(0,1)$-form to be in the domain of an actual adjoint of $\bar\partial$ involves the vanishing of its normal components on boundary and is usually applied to give the vanishing of $\Phi(z_1)$ on ∂D.

$\bar\partial$ Estimates for Complex Algebraic Manifold from Arguments on Riemann Domain over $\mathbb{C}^n$ after Removing Tubular Neighborhood of Ample Complex Hyersurface. The Cauchy integral formula for smooth functions is for domains in $\mathbb{C}$. Since the $\bar\partial$ estimates for a complex algebraic manifold can be derived from arguments on a Riemann domain over $\mathbb{C}^n$ after removing a tubular neighborhood of an ample complex hyersurface, with appropriate modification we can still apply the Cauchy integral formula for the inverse image of every complex line in $\mathbb{C}^n$.

An Example and Illustration of the Use of Gradient Term. As an example and an illustration of the use of the gradient term, I wrote up the following.

Yum-Tong Siu, The Role of the Gradient Term of the Bochner-Kodaira Formula in Coherent Sheaf Extension. Sibony Memorial Issue of the Journal of Geometric Analysis (2022). https://arXiv.org/pdf/2112.12894.pdf, https://doi.org/10.1007/s12220-022-00993-1

The result is as follows. Let X be a complex manifold and Y be an irreducible complex hypersurface in X and U be some closed coordinate ball in X of positive radius centered at some point of Y. Let E be a holomorphic vector bundle on $(X-Y)\cup U$ with a smooth Hermitian metric whose curvature has a finite global L^p norm for some $p>1$. Then the locally free sheaf $\mathcal{O}_X(E)$ can be uniquely extended to a reflexive sheaf on X.

It is obtained by applying the Bochner-Kodaira formula on the product of an annulus and a polydisk to the produce an annulus and a polydisk. The tools are only the Cauchy integral formula for smooth functions and Hölder inequalities, applied repeatedly.

Recently Song Sun and Xuemiao Chen told me that they read my paper carefully and there is one step they could not follow. The formal presentation in the paper is simplified from a more involved argument. Very likely there is some oversimplification and some of the original arguments need to be put back. I will look into it and make available any necessary modifications.

Part II. *New Techniques of Higher-Order Jets and Eigenfunctions in Global Nondeformability.* Kodaira and Spencer posed in their 1958 Ann. of Math. paper (p.464, Problem 8) whether $\mathbb{P}_n$ admits any global holomorphic deformation.

K. Kodaira and D. C. Spencer, On Deformations of Complex Analytic Structures, II. Ann. of Math. 67 (1958), 403 – 466.

More precisely, whether there exists a holomorphic family $\pi:\mathcal{X}\to\mathbb{D}$ of compact complex manifolds over the open unit 1-disk $\mathbb{D}$ such that $\pi^{-1}(t)=\mathbb{P}_n$ for $t\neq 0$ without the center fiber $\pi^{-1}(0)$ biholomorphic to $\mathbb{P}_n$.

Case of $\mathbb{P}_n$. The problem of Kodaira and Spencer was solved with the affirmative answer of global nondeformability of $\mathbb{P}_n$ in

Yum-Tong Siu, Nondeformability of the complex projective space. J. Reine Angew. Math. 399 (1989), 208 – 219. Errata. *ibid.* 431 (1992), 65 – 74.

A natural follow-up problem is whether any irreducible compact Hermitian symmetric manifold is globally nondeformable.

Global Nondeformability of Irreducible Compact Hermitian Symmetric Manifold. Let Y be an irreducible compact Hermitian symmetric manifold. Let $\pi : X \to \mathbb{D}$ is a holomorphic family of compact complex manifolds over the open unit 1-disk $\mathbb{D}$ such that $X_t := \pi^{-1}(t)$ is biholomorphic to Y for $t \neq 0$ in $\mathbb{D}$. The problem is to prove that $X_0 := \pi^{-1}(0)$ must also be biholomorphic to Y.

The case of the hyperquadric Q_n in $\mathbb{P}_{n+1}$ is confirmed in

Jun-Muk Hwang, Nondeformability of the complex hyperquadric. Invent. Math. 120 (1995), 317 – 338.

However, the case of a general irreducible compact Hermitian symmetric manifold (other than $\mathbb{P}_n$ and Q_n) remains open.

Kähler Nondeformability. Under the additional assumption that the center fiber is Kähler, the global nondeformability of irreducible compact Hermitian symmetric manifolds, together with other related developments, is proved by Jun-Muk Hwang and Ngaiming Mok in a very important series of papers on the theory of varieties of minimal rational curves.

Jun-Muk Hwang and Ngaiming Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. Invent. Math. 131 (1998), 393 – 418.

Minimal Rational Curves in General Fiber and Possible Degeneracy of Criss-Crossing Property of Their Limits in Center Fiber. When minimal rational curves in a general fiber approach the center fiber, they may decompose into several branches of rational curves. If enough remain irreducible and retain the criss-crossing property, one can verify that the center fiber is Hermitian symmetric. When there is a Kähler metric in the deformation, the minimal rational curves from a general fiber would not decompose into branches in the limit.

We now discuss our new technique without the Kähler assumption.

Linear Independence of Sections for General Fiber Persists in Limit Fiber. Let $\pi : X \to \mathbb{D}$ be a holomorphic family of compact complex manifolds whose fibers $X_t = \pi^{-1}(t)$ are biholomorphic for $t \neq 0$. Let L be a holomorphic line bundle on X whose restriction to X_t is denoted by L_t. Assume that

$\dim_{\mathbb{C}} \Gamma(X_t, L_t) = N$ for $t \neq 0$. Then there exist N holomorphic section $s_1, \cdots, s_N$ of L over X such that for every $t \in \mathbb{D}$ (including $t = 0$), the holomorphic sections $s_1|_{X_t}, \cdots, s_N|_{X_t}$ are $\mathbb{C}$-linearly independent.

The reason is that if $s_N = \sum_{j=1}^{N-1} c_j s_j$ on X_0 for some complex numbers $c_1, \cdots, c_{N-1}$, then

$$s = \frac{1}{t}\left(s_N - \sum_{j=1}^{N-1} c_j s_j\right).$$

is in $\Gamma(X, \mathcal{O}_X(L))$ and one can repeat the process of dividing, by appropriate powers of t, certain linear combinations fiberwise generators $s_1, \cdots, s_N$ outside the center fiber, when $t \to 0$. We refer to this process as renormalization by appropriate powers of t.

Another way to look at it is that it is the result of the torsion-freeness of the direct image sheaf $R^0\pi_*\mathcal{O}(L)$ on $\mathbb{D}$.

We can replace $\mathcal{O}(L)$ by another coherent sheaf $\mathcal{F}$, for example, some appropriately ideal sheaf of $\mathcal{O}(L_t)$ for each $t \neq 0$, biholomorphical the same for all $t \neq 0$,

Linear Independence of Sections on Limit Fiber Very Different from Independence of Their Differentials. When L_t is ample (or even very ample), though $s_1, \cdots, s_N$ on X_0 are $\mathbb{C}$-linearly independent, we cannot draw any conclusion about the number of $\mathbb{C}$-linear independent $ds_1, \cdots, ds_N$ locally at some point of X_0.

However, for $m \in \mathbb{N}$ sufficiently large, when we replace L by mL, with $N_m + 1 = \dim \Gamma(X_t, mL_t)$ for $t \neq 0$, we can say more about the N_m holomorphic sections $\sigma_0, \cdots, \sigma_{N_m}$ of mL over X with $\sigma_0|_{X_t}, \cdots, \sigma_{N_m}|_{X_t}$ $\mathbb{C}$-linearly independent for every $t \in \mathbb{D}$ (including $t = 0$). What we can say more is that, at every point P outside a proper subvariety Z_0 of X_0, we can select $n+1$ of $\sigma_0|_{X_0}, \cdots, \sigma_{N_m}|_{X_0}$ to serve as local projective coordinates at points near P. This kind of arguments is like showing the limit of algebraric manifolds to be Moishezon.

We are going to use these local projective coordinates as a frame of reference to guarantee the existence of the limit of certain holomorphic vector fields like Euler vector fields without renormalization by powers of t.

By Hartogs' extension theorem, as long as we have certain holomorphic vector fields like Euler vector fields on the limit fiber without renormalization by powers of t at some point of the limit fiber, we automatically have their holomorphic extension on all of the limit fiber.

What is really new in our approach is to use eigenfunctions of differential operators from iterated action Euler vector fields in terms of the frame of reference described above.

It is easier to explain the procedure by first going over the $\mathbb{P}_n$ case.

For $Y = \mathbb{P}_n$, choose $L = \mathcal{O}_{\mathbb{P}_n}(1)$ and get a $\mathbb{C}$-basis $s_0, \cdots, s_n$ of global sections.

An Euler vector field for Y depends on a choice of the infinity hyperplane $\mathbb{P}_{n-1}^{\infty}$ and the origin. It can be written as $\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$ when $z_j = \frac{s_j}{s_0}$ for $j \neq 0$ are the affine coordinates.

It means that the infinity hyperplane is chosen to be $\{s_0 = 0\}$ and the origin is the point $s_1 = \cdots = s_n = 0$.

For the family $\pi : X \to \mathbb{D}$, we have elements of $s_0, \cdots, s_n$ of $\Gamma(X, L)$ and we choose the holomorphic section of origins so that the origin at X_0 is outside some prescribed proper subvariety Z_0 of X_0.

For m sufficiently large, we have elements of elements of $\sigma_0, \cdots, \sigma_{N_m}$ of $\Gamma(X, mL)$ with $N_m + 1 = \dim \Gamma(X_t, mL_t)$ for $t \neq 0$. We have the subvariety Z_0. One important point is that on X_t for $t \neq 0$, each σ_j is a homogeneous polynomial of $s_0, \cdots, s_n$ of degree m.

However, we have no such information on X_0, because $s_0, \cdots, s_n$ (and $\sigma_0, \cdots, \sigma_{N_m}$) on X_0 are constructed from the process of using renormalization by appropriate powers of t at each step of the process (which is equivalent to using Nakayama's lemma).

Vector Fields of Projection onto Eigenfunctions. In the case of $\mathbb{P}_n$ consider iterated action of the Euler vector field

$$\vec{v} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

on X_t for $t \neq 0$, which acts on

$$\mathcal{P} = \sum_{k=1}^{m} \mathcal{P}_k,$$

where $\mathcal{P}_k$ is homogeneous of degree m in $s_0, \cdots, s_n$, to give

$$(\%) \qquad \vec{v}^{\ell}\, \mathcal{P} = \sum_{k=1}^{m} k^{\ell}\, \mathcal{P}_k \quad \text{for} \quad 1 \leq \ell \leq m.$$

We apply Cramers' rule to solve for the m unknowns $\mathcal{P}_1, \cdots, \mathcal{P}_m$ in the system of m linear equations in (%). Let

$$A = \det\left(k^{\ell}\right)_{1 \leq k, \ell \leq m} = \det \begin{pmatrix} 1 & 1^2 & \cdots & 1^m \\ 2 & 2^2 & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots \\ m & m^2 & \cdots & m^m \end{pmatrix}.$$

Its value can be computed from the Vandemonde determinant

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{pmatrix} = \prod_{1 \leq k < \ell \leq m} (x_\ell - x_k),$$

with $x_1 = 1, x_2 = 2, \cdots, x_m = m$ after we divide A by $1 \cdot 2 \cdot 3 \cdots m$ (*i.e.*, divide the j-th row of the matrix of A by j for $1 \leq j \leq m$) to get

$$A = m! \prod_{1 \leq k < \ell \leq m} (\ell - k),$$

which is nonzero. Cramer's rule yields

$$A\mathcal{P}_k = \det \begin{pmatrix} 1 & 1^2 & \cdots & 1^{k-1} & \vec{v}\,\mathcal{P} & 1^{k+1} & \cdots & 1^m \\ 2 & 2^2 & \cdots & 2^{k-1} & \vec{v}^2\,\mathcal{P} & 2^{k+1} & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m & m^2 & \cdots & m^{k-1} & \vec{v}^m\,\mathcal{P} & m^{k+1} & \cdots & m^m \end{pmatrix}.$$

It follows that

$$
\begin{aligned}
\mathcal{P} &= \sum_{k=1}^{m} \mathcal{P}_k \\
&= \frac{1}{A} \sum_{k=1}^{m} \det \begin{pmatrix}
1 & 1^2 & \cdots & 1^{k-1} & \vec{v}\,\mathcal{P} & 1^{k+1} & \cdots & 1^m \\
2 & 2^2 & \cdots & 2^{k-1} & \vec{v}^2\,\mathcal{P} & 2^{k+1} & \cdots & 2^m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m & m^2 & \cdots & m^{k-1} & \vec{v}^m\,\mathcal{P} & m^{k+1} & \cdots & m^m
\end{pmatrix}.
\end{aligned}
$$

Expanding the k-th term of he right-hand side according to its k-th column, we get

$$\sum_{\ell=1}^{m} B_\ell \vec{v}^\ell\,\mathcal{P}.$$

Let

$$\vec{\xi}^{(m)} = \sum_{\ell=1}^{m} B_\ell \vec{v}^\ell$$

be the m-jet differentiation obtained as a $\mathbb{C}$-linear combination of the ℓ-th iterated action of $\vec{v}$ for $1 \leq \ell \leq m$. Each polynomial $\mathcal{P}$ of degree $\leq m$ in $s_0, \cdots, s_m$ is an eigenfunction for $\vec{\xi}^{(m)}$. We can now conclude that $\vec{\xi}^{(m)}$ extends (without modification by a factor of a power of t) to all of

$$X = \bigcup_{t \in \mathbb{D}} X_t$$

including the center fiber, because we can for a generic point of X_0, choose n such polynomials $\mathcal{P}$ as local coordinates of X_0.

Note that at X_t with $t \neq 0$, the coefficients of $\mathcal{P}$ in general depend on t when the polynomials $\mathcal{P}$ for each X_t for $t \neq 0$ are chosen to be pieced together to form a basis of the locally free zeroth direct image on $\mathbb{D}$, as explained above in the procedure of renormalization by an appropriate power of t. In particular, the Euler vector field as one of the m terms in $\vec{\xi}^{(m)}$ can be extended (without modification by a factor of a power of t) to all of

$$X = \bigcup_{t \in \mathbb{D}} X_t$$

including the center fiber.

We then can use the Lie algebra structure of holomorphic vector fields to conclude that the center fiber is compact Hermitian symmetric.

Euler Vector Fields in Hyperquadrics. The Euler vector field for Q_n is constructed from the tautological Euclidean space $\mathbb{C}^{n+2}$ (with coordinates $\zeta_0, \cdots, \zeta_{n+1}$) of $\mathbb{P}_{n+1}$ where Q_n lies as a complex hypersurface of degree 2.

The inverse image $\tilde{Q}_n$ of Q_n in $\mathbb{C}^{n+2}$ is defined by

$$\zeta_0\zeta_1 + \zeta_2\zeta_3 + \cdots + \zeta_n\zeta_{n+1} = 0$$

(under the assumption that $n = 2m$ is even), which means that if $z_j = \frac{\zeta_j}{\zeta_0}$ denotes the j-th component of the affine coordinate of $\mathbb{P}_{n+1} \cap \{\zeta_0 \neq 0\}$, then Q_n is defined on the affine part of $\mathbb{P}_{n+1} \cap \{\zeta_0 \neq 0\}$ of $\mathbb{P}_n$ by

$$z_1 + z_2z_3 + \cdots + z_nz_{n+1} = 0.$$

A vector field in $\mathbb{C}^{n+1}$ which is tangential to Q_n is

$$\vec{v} = 2z_1\frac{\partial}{\partial z_1} + \sum_{j=1}^{2n+1} z_j\frac{\partial}{\partial z_j}.$$

These are all Euler vector fields in the sense that they give $\mathbb{C}^*$-action and we know how to construct their eigenfunctions. An important point is that the integral curves may not be minimal rational curves. For example, $z_1 = -\tau^2$ and $z_2 = z_3 = \tau$ parametrized by $\tau \in \mathbb{C}$ would be an integral curve. The minimal rational curves are like $z_2 = cz_4$ and $cz_3 = z_5$ for any $c \in \mathbb{C}$ with $z_j = 0$ for $j \neq 2, 3, 4, 5$.

Eigenfunctions for Euler Vector Fields in Hyperquadrics. The eigenfunctions for $\vec{v}$ on Q_n are the restrictions of polynomials on the affine part $\mathbb{C}^{n+1}$ which are homogeneous in the variables $z_1, z_2^2, \cdots, z_{2n+1}^2$ (*i.e.,* with homogeneous weight $1, 2, \cdots, 2$), because

$$\begin{aligned}&\vec{v}\left(z_1^{\ell_1}(z_2^2)^{\ell_2}\cdots(z_{2n+1}^2)^{\ell_{2n+1}}\right)\\&= (2\ell_1 + 2\ell_2 + \cdots + 2\ell_{2n+1})\left(z_1^{\ell_1}(z_2^2)^{\ell_2}\cdots(z_{2n+1}^2)^{\ell_{2n+1}}\right).\end{aligned}$$

Then we proceed as in the $\mathbb{P}_n$ case.

Grassmannians are more complicated but along the same lines. In the Kuranishi memorial conference in May 2022 I talked also about the global nondeformability problem for the Grassmannian. In the posted PDF file from that conference you can find more precise discussions and formulas. So here I will not go further into the global nondeformability problem for the Grassmannian.

Before going to the third topic, I would like to make two remarks. The first remark is that to solve the hyperbolicity problem of complex hypersurfaces of sufficiently high degree, I already used jets of higher order, with motivation from Herb Clemens's unusually innovative methods for its analogue in the context of algebraic geometry. The second remark is that our new technique to prove global nondeformability of Grassmannians without Kähler assumption does not in any way detract from the importance of the theory of varieties of minimal rational curves developed by Jun-Muk Hwang and Ngaiming Mok. As a matter of fact, the 2022 Future Science Prize in Mathematics and Computer Science (considered the Nobel Prize of China) was awarded to Ngaiming Mok with the theory of varieties of minimal rational curves mentioned as one of his two contributions in the award citation.

Part III. *Stabilization of Irreducible Components of Lelong Sets in Problems of Finite Generation and the Abundance Conjecture.*

Analysis is the most natural and the simplest platform for the treatment of finite generation and the abundance conjecture (for the case of nonnegative numerical Kodaira dimension). Some of the tools are the following.

(i) *Global Generation of Multiplier Ideal Sheaves.* Let Y be a compact complex manifold of complex dimension n and L be a holomorphic line bundle over Y with metric $e^{-\xi}$ of semipositve curvature current. Let E be an ample holomorphic line bundle over Y such that for every point P of Y there are a finite number of elements of $\Gamma(Y, E)$ which all vanish to order at least $n+1$ at P and which do not simultaneously vanish outside P. Then $\Gamma(Y, \mathcal{I}_\xi \otimes (L+E+K_Y))$ generates $\mathcal{I}_\xi \otimes (L+E+K_Y)$ at every point of Y.

This particular tool of the global generation of multiplier ideal sheaves is useful only in the case of general type.

(ii) *Lelong Numbers and Lelong Sets for Closed Positive* $(1,1)$*-Currents and Plurisubharmonic Functions.* A closed positive $(1,1)$-current Θ on some open ball neighborhood U of 0 in $\mathbb{C}^n$ means

$$\Theta = \frac{i}{2\pi}\partial\bar{\partial}\varphi$$

for some plurisubharmonic function φ on U. On the restriction of Θ to a generic complex line L through 0, the restriction of Θ to L has a point mass of magnitude γ at 0 and this common value γ is the Lelong number of Θ at 0. The set E_c of points P in U where the Lelong number of Θ is $\geq c$ is a complex-analytic subvariety. When Θ is defined by the integration of $(n-1,n-1)$-forms over the regular part of a complex hypersurface Y, the Lelong number of Θ at a point P of Y is equal to the multiplicity of Y at P. When $\varphi = \log\Phi$ with

$$\Phi = \sum_m \varepsilon_m \left|f_m\right|^2$$

being a convergent infinite sum of absolute value-squares of multi-valued holomorphic functions f_m (where $\varepsilon_m > 0$ and $f_m^{N_m}$ is a holomorphic function for some $N_m \in \mathbb{N}$), the Lelong number of Θ at P is the minimum vanishing order of $|f_m|^2$ at P (over m) along a generic complex line L through P.

(iii) *Poisson-Jensen Formula for Currents Linking Lelong Number of* $\frac{i}{2\pi}\partial\bar{\partial}\varphi$ *to Integrability of* $e^{-\varphi}$. For $a \in \mathbb{C}^n$ the Poisson-Jensen formula for currents

$$\varphi(a) = \int_{|z-a|=R} \varphi\,\omega_a^{n-1} + \frac{i}{\pi}\int_{|z-a|<R}\left(\log\frac{|z-a|}{R}\right)\partial\bar{\partial}\varphi\wedge\omega_a^{n-1},$$

where

$$\omega_a = \frac{i}{2\pi}\,\partial\bar{\partial}\log|z-a|^2,$$

is obtained from the Green identity on a complex line L through a (involving the Laplacians of φ and $\log|z-a|^2$) and then averaging over L. This kind of formula is used in Nevanlinna theory.

By using the convexity of the exponential function, one can use it to obtain Skoda's result on the integrability of $e^{-\varphi}$ when when the Lelong number of Θ_φ is < 1. More precisely, just as in Nevanlinna theory of carrying out once more integration in the radial direction, for $R_1 < R < R_2$,

$$\varphi(a) = \frac{1}{R_2 - R_1}\int_{R_1<|z-a|<R_2} \varphi\,\omega_a^{n-1} + \frac{1}{R_2 - R_1}\int_{R_1<R<R_2}\left(\int_{|z-a|<R}\left(\log\frac{|z-a|^2}{R^2}\right)\Theta_\varphi\wedge\omega_a^{n-1}\right)dR$$
$$= \mathcal{A}_{a,R_1,R_2}(\varphi\,\omega_a^{n-1}) + \mathcal{J}_{a,R_1,R_2}\left(\left(\log\frac{|z-a|^2}{R^2}\right)\Theta_\varphi\wedge\omega_a^{n-1}\right),$$

where

$$\mathcal{A}_{a,R_1,R_2}(u) = \frac{1}{R_2 - R_1}\int_{R_1<|z-a|<R_2} u$$

and

$$\mathcal{J}_{a,R_1,R_2}(u) = \frac{1}{R_2 - R_1}\int_{R_1<|z-a|<R_2}\left(\int_{|z-a|<R} u\right)dR.$$

If the Lelong number of its curvature current

$$\Theta_\varphi = \frac{i}{2\pi}\partial\bar\partial\varphi$$

is < 1 at 0, then for R sufficiently small and for a sufficiently small neighborhood W of 0 in U

$$\int_{|z-a|<R}\Theta_\varphi\wedge\left(\frac{i}{2\pi}\,\partial\bar\partial\log|z-a|^2\right)^{n-1}$$

is $< 1-\varepsilon$ for $a\in W$ and for some $0<\varepsilon<1$. Thus,

$$\int_{a\in W} e^{-\varphi(a)} \leq \left\{\sup_{a\in W}\left[\exp\left(-\mathcal{A}_{a,R_1,R_2}(\varphi\omega_a^{n-1}) + \mathcal{J}_{a,R_1,R_2}\left((2\log R)\Theta_\varphi\wedge\omega_a^{n-1}\right)\right)\right]\right\}\cdot$$
$$\cdot\left\{\int_{a\in W}\int_{|z-a|<R}\frac{1}{|z-a|^{2(1-\varepsilon)}}\right\}\left\{\sup_{a\in W}\left(\frac{\Theta_\varphi\wedge\omega_a^{n-1}}{\mathcal{J}_{a,R_1,R_2}\left(\Theta_\varphi\wedge\omega_a^{n-1}\right)}\right)\right\}$$

is finite.

I include the technical details in this step so that research workers in Nevanlinna theory can see how the theory of Lelong numbers is related to the techniques in Nevanlinna theory which they are familiar with.

The argument can be applied also to the difference of two plurisubharmonic functions as follows.

Let $n \geq 2$ and $R_0 > 0$ and $B_{R_0}(0)$ be the open ball in $\mathbb{C}^n$ of radius R_0 centered at the origin 0. Let φ and ψ be plurisubharmonic functions on $B_{R_0}(0)$. Suppose the Lelong numbers of both their $(1,1)$-closed positive currents

$$\Theta_\varphi = \frac{i}{2\pi}\,\partial\bar\partial\varphi \quad \text{and} \quad \Theta_\psi = \frac{i}{2\pi}\,\partial\bar\partial\psi$$

are zero at every point of $B_{R_0}(0) - \{0\}$. Moreover, assume that the Lelong number of Θ_φ at 0 is less than 1 plus the Lelong number of Θ_ψ at 0. Then $e^{-(\varphi-\psi)}$ is locally integrable at every point of $B_{R_0}(0)$.

(iv) *Stabilization of Irreducible Components of Lelong Sets with Finite Truncation of Infinite Sum of Absolute-Value Squares of Multi-Valued Holomorphic Functions.* When

$$\Phi = \sum_m \varepsilon_m \left|f_m\right|^2$$

is truncated to a finite sum

$$\Phi_k = \sum_{m \leq k} \varepsilon_m \left|f_m\right|^2,$$

the Lelong sets decrease as k increases. When irreducible components of Lelong sets are considered, the decrease may go to a lower dimension, but the situation stabilizes with some finite truncation.

(v) *Skoda's Ideal Generation.* The original statement is as follows. Let Ω be a domain spread over $\mathbb{C}^n$ which is Stein. Let ψ be a plurisubharmonic function on Ω, $g_1, \ldots, g_p$ be holomorphic functions on Ω, $\alpha > 1$, $q = \min(n, p-1)$, and f be a holomorphic function on Ω. Assume that

$$\int_\Omega \frac{\left|f\right|^2 e^{-\psi}}{\left(\sum_{j=1}^p \left|g_j\right|^2\right)^{\alpha q+1}} < \infty.$$

Then there exist holomorphic functions $h_1, \ldots, h_p$ on Ω with $f = \sum_{j=1}^p h_j g_j$ on Ω such that

$$\int_\Omega \frac{\left|h_k\right|^2 e^{-\psi}}{\left(\sum_{j=1}^p \left|g_j\right|^2\right)^{\alpha q}} \leq \frac{\alpha}{\alpha-1} \int_\Omega \frac{\left|f\right|^2 e^{-\psi}}{\left(\sum_{j=1}^p \left|g_j\right|^2\right)^{\alpha q+1}}$$

for $1 \leq k \leq p$.

Henri Skoda, Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids. *Ann. Sci. École Norm. Sup.* **5** (1972), 545-579.

Reformation to suit our purpose is as follows. Let X be a compact complex algebraic manifold of complex dimension n, L be a holomorphic line bundle over X, and E be a holomorphic line bundle on X with metric $e^{-\psi}$ such that ψ is plurisubharmonic. Let $k \geq 1$ be an integer, $G_1, \ldots, G_p \in \Gamma(X, L)$, and $|G|^2 = \sum_{j=1}^{p} |G_j|^2$. Let $\mathcal{I} = \mathcal{I}_{(n+k+1)\log|G|^2+\psi}$ and $\mathcal{J} = \mathcal{I}_{(n+k)\log|G|^2+\psi}$. Then

$$\begin{aligned}&\Gamma\left(X, \mathcal{I} \otimes ((n+k+1)L + E + K_X)\right) \\ &= \sum_{j=1}^{p} G_j \, \Gamma\left(X, \mathcal{J} \otimes ((n+k)L + E + K_X)\right).\end{aligned}$$

All Tools Put Together to Conclude Finite Generation of Canonical Ring. The arguments of putting together the above tools are quite natural in concluding the finite generation of the canonical ring for a compact complex algebraic manifold X.

Reduction to General Type. The analytic approach allows the following reduction process to go to a quotient of general type.

If X is not of general type, we let $s_j^{(m)}$ for $1 \leq j \leq q_m$ be a $\mathbb{C}$-basis of $\Gamma(X, mK_X)$ and consider the construction of a quotient space of X which is of general type. The relation subspace in $X \times X$ is given by the Lelong sets defined by the limit of

$$\frac{1}{m} \log \sum_{j,k=1}^{q_m} \left| s_k^{(m)}(x) s_\ell^{(m)}(y) - s_\ell^{(m)}(x) s_k^{(m)}(y) \right|^2$$

as $m \to \infty$ on $X \times X$ as a function of $(x, y) \in X \times X$. This means that we are interested in the stable situation when $s_j^{(m)}$ for $1 \leq j \leq q_m$ are used local projective coordinates.

This reduction technique also allows an additional fixed twisted by some ample line bundle A of X as follows.

Let $s_{A,j}^{(m)}$ for $1 \leq j \leq q_{A,m}$ be a $\mathbb{C}$-basis of $\Gamma(X, mK_X + A)$ and consider the construction of a quotient space of X by using the relation subspace in $X \times X$ which is given by the Lelong sets defined by the limit of

$$\frac{1}{m} \log \sum_{j,k=1}^{q_{A,m}} \left| s_{A,k}^{(m)}(x) s_{A,\ell}^{(m)}(y) - s_{A,\ell}^{(m)}(x) s_{A,k}^{(m)}(y) \right|^2$$

as $m \to \infty$ on $X \times X$ as a function of $(x, y) \in X \times X$.

This technique of passing to limit as $m \to \infty$ for a fixed ample line bundle A in the consideration of $\Gamma(X, mK_X + A)$ is useful in treating the following abundance conjecture in the case when the numerical Kodaira dimension is assumed nonnegative, which means that we actually do have sections to consider.

Abundance Conjecture. For a compact complex algebraic manifold X of complex dimension n, we define its Kodaira dimension

$$\kappa_{\rm kod}(X) = \limsup_{m \to \infty} \frac{\log \dim_{\mathbb{C}} \Gamma\left(X, mK_X\right)}{\log m}$$

and its numerical Kodaira dimension

$$\kappa_{\rm num}(X) = \sup_{k \geq 1} \left[\limsup_{m \to \infty} \frac{\log \dim_{\mathbb{C}} \Gamma\left(X, mK_X + kA\right)}{\log m} \right],$$

where A is any ample line bundle on X. The definition of $\kappa_{\rm num}(X)$ is independent of the choice of the ample line bundle A on X.

We use the analytic viewpoint and formulate the abundance conjecture as $\kappa_{\rm kod}(X) = \kappa_{\rm num}(X)$. Roughly speaking, it means that the γ in the growth order m^γ of $\dim_{\mathbb{C}} \Gamma\left(X, mK_X + A\right)$ as $m \to \infty$ is independent of γ when the ample twisting by A (or kA) is fixed.