Complex Differential Geometry in the Solution of Arithmetico-Geometric Problems over Complex Function Fields

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Moduli Space of Elliptic Curves

An elliptic curve is complex-analytically a compact Riemann surface S of genus 1. In other words, $S := \mathbb{C}/L$ **for some lattice** $L \subset \mathbb{C}$. Replacing L by λL for some $\lambda \in \mathbb{C} - \{0\}$, without loss of generality we may assume $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$, i.e., $\tau \in \mathcal{H}$, where $\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, the upper half plane. Write $S_\tau := \mathbb{C}/L_\tau$.

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For $\tau, \tau' \in \mathcal{H}$, we have $S_\tau \cong S_{\tau'}$ **if and only** if there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$ such that $L_{\tau'} = \lambda L_\tau$, **i.e.**, if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ **where** $ad - bc \neq 0$. **Thus, the set of equivalence classes of $\mathbb{C}/L$ is in one-to-one correspondence with $X = X(1) := \mathcal{H}/\mathbb{P}SL(2,\mathbb{Z})$.** $\mathbb{P}SL(2,\mathbb{Z})$ acts discretely on $\mathcal{H}$ with fixed points. **We have the j-function $j : X(1) \xrightarrow{\cong} \mathbb{C}$, and $\overline{X(1)} = \mathbb{P}^1$.**

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A suitable finite-index subgroup $\Gamma \subset \mathbb{P}SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ without fixed points and $X_\Gamma := \mathcal{H}/\Gamma$ can be compactified to a compact Riemann surface.

The j-function

On the upper half plane $\mathcal{H} = \{\tau : \mathrm{Im}(\tau) > 0\}$ define

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)}$$

where $g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-4}$; $g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-6}$.

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The j-function establishes a biholomorphism $j : \mathcal{H}/\mathrm{SL}(2,\mathbb{Z}) \xrightarrow{\cong} \mathbb{C}$.

Invariant Kähler metrics on $\mathcal{H} \times \mathbb{C}$

On $\pi : \mathcal{H} \times \mathbb{C} \to \mathcal{H}$, there is the relative tangent bundle $V = T_\pi$, and the horizontal real-analytic integrable subbundle $H \subset T(\mathcal{H} \times \mathbb{C})$ whose leaves are images of **horizontal** sections $w = a + b\tau$, $a, b \in \mathbb{R}$. We have $T(\mathcal{H} \times \mathbb{C}) = V \oplus H$. There is a semi-Kähler form μ with kernel H so that, denoting by ω the Kähler form of the Poincaré metric on $\mathcal{H}$, and defining $\nu_t := \pi^*\omega + t^2\mu$, $t > 0$, $(\mathcal{H} \times \mathbb{C}, \nu_t)$ is a Kähler form invariant under $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a torsion-free finite index subgroup. Write $X^0_\Gamma := \mathcal{H}/\Gamma$, $\mathcal{M}^0_\Gamma = (\mathcal{H} \times \mathbb{C})/(\Gamma \ltimes \mathbb{Z}^2)$, $\pi : \mathcal{M}_\Gamma \to X_\Gamma$ a compactification to a minimal elliptic surface over the projective curve X_Γ.

Verticality of a section

Let $\sigma : X_\Gamma \to \mathcal{M}_\Gamma$ be a holomorphic section and $d\sigma : TX_\Gamma \to \sigma^* T(\mathcal{M}_\Gamma)$ be its differential. Define the *verticality* of σ as $\eta_\sigma := \Pi_V \circ d\sigma|_{T(X^0_\Gamma)} : T(X^0_\Gamma) \to \sigma^* V$. Thus, η_σ is a real-analytic section of the holomorphic line bundle $T^*(X^0_\Gamma) \otimes \sigma^* V$ on X^0_Γ.

Shioda's Theorem: A differential-geometric proof

Proposition (geometric characterization of torsion sections)

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Proof: Given a holomorphic section $\sigma : X_\Gamma \to \mathcal{M}_\Gamma$ σ corresponds to $f : \mathcal{H} \to \mathbb{C}$ satisfying $f(\gamma\tau) = \frac{f(\tau)}{c_\gamma\tau + d_\gamma} + A_\gamma(\gamma\tau) + B_\gamma$ for some integers A_γ, B_γ, in which $\gamma(\tau) = \frac{a_\gamma\tau + b_\gamma}{c_\gamma + d_\gamma}$. Then, $f''(\gamma\tau) = (c_\gamma\tau + d_\gamma)^3 f''(\tau)$. (Eichler) We discovered that $\xi_\sigma := f''(\tau)(d\tau)^{\frac{3}{2}}$ is actually given by $\xi_\sigma = \nabla\eta_\sigma$. We have $\overline{\partial}\xi_\sigma = 0$, hence $\overline{\partial}\nabla\eta_\sigma = 0$. Interchanging the order of differention we have $\overline{\nabla}^*\overline{\nabla}\eta_\sigma = -\eta_\sigma$. Integrating by parts we get $\int_{X_\Gamma} \|\eta_\sigma\|^2\omega = -\int_{X_\Gamma} \|\overline{\nabla}\eta_\sigma\|^2\omega$, forcing $\eta_\sigma \equiv 0$, hence σ is a torsion section.

Betti coordinates and the Betti map of a section

Betti coordinates

On $\mathcal{H} \times \mathbb{C}$, for a point (τ, w), express w in terms of a basis of the lattice L_τ, e.g., $w = \beta_1 \cdot 1 + \beta_2 \tau$. The pair (β_1, β_2) are Betti coordinates.

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The Betti map associated to a holomorphic section σ

For a holomorphic section $\sigma : X_\Gamma \to \mathcal{M}_\Gamma$, the local pullback $\beta := (\sigma^*\beta_1, \sigma^*\beta_2)$ is called the *Betti map* of σ. Since the construction of (β_1, β_2) involves a choice of abelian logarithm on $\mathcal{M}_\Gamma^0$, so does the Betti map β, but the vanishing order of β at any point $b \in B^0$ is independent of such choice and is intrinsic to the section σ.

The Betti map

The following definition is due to Corvaja-Demeio-Masser-Zannier.

The Betti multiplicity of a Betti map at a finite point

The *multiplicity* of a Betti map β at b is defined to be the smallest positive integer $m(b)$ such that the partial derivatives of $\sigma^*\beta_1, \sigma^*\beta_2$ at b vanish up to order $m(b) - 1$. We will also call $m(b)$ the Betti multiplicity of σ at b.

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The Betti multiplicity of a Betti map at a cusp

When a holomorphic section σ cuts over a base point c of bad reduction, i.e., corresponding to a cusp, we express the section σ locally near the cusp c in terms of toroidal compactification $\Sigma(w) = (\xi(w), \zeta(w))$ If $|\xi(0)| = 1$, then we define the Betti multiplicity m_c of σ at c to be the vanishing order of $\xi(w) - \xi(0)$ at $w = 0$. Otherwise, we define $m_c = 1$.

Betti Multiplicities for a Section of an Elliptic Surface

Theorem (Ulmer-Ursúa *IMRN* 2021)

Suppose $\pi : \mathcal{E} \to B$ is a non-isotrivial minimal elliptic surface, with exactly δ singular fibers, and $\sigma : B \to \mathcal{E}$ be a section of infinite order. Denote by g be the genus of B. Let O denote the zero section of $\mathcal{E}$ and denote by d the degree of the holomorphic line bundle $O^*\Omega^1_{\mathcal{E}|C}$, where $\Omega^1_{\mathcal{E}|C}$ denotes the dual of the relative tangent bundle. Denote by $S \subset B$ the set of base points of singular fibers, and write $B^0 := B - S$. Then,

$$\sum_{b \in B^0} (m_b - 1) \leq 2g - 2 - d + \delta\,.$$

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(a) The finiteness of points of B^0 with multiplicities ≥ 2 was due to Corvaja-Demeio-Masser-Zannier (*Crelles* 2022)

(b) Multiplities m_c at cusps were defined algebraically and using the Kodaira classification of elliptic surfaces, and the analytic definition of Mok-Ng using toroidal coordinates agree with the algebraic definition. **Equality was proven when the sum on the left hand side is replaced by taking all $b \in B$, including the cusps.**

Diff.-geom. proof for estimates on Betti multiplicities

Theorem (Mok-Ng 2022)

Let $\mathcal{E} \to B$ be an elliptic surface over a projective curve B with a classifying map $f : B \to X$ of degree d, where $X = X_{\Gamma(k)}$ for some $k \geq 3$. Let σ be a non-torsion section of $\mathcal{E}$ and m_b be the Betti multiplicity of σ at b, then

$$\sum_{b \in B} (m_b - 1) = \sum_{b \in B \setminus S} (r_b - 1) + \frac{d}{2\pi} \int_{X^0} \omega,$$

where $X^0 = X^0_{\Gamma(k)}$ and $S = f^{-1}(X \setminus X^0)$; r_b is the ramification index of f at b and ω is the Kähler form on X^0 descending from the invariant form $-i\partial\bar{\partial} \log \mathrm{Im}\tau$ on $\mathcal{H}$.

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Corollary

Denote by $\mathfrak{B}_\sigma$ the divisor of points on B^0 over which the Betti multiplicity $m_b \geq 2$, with weight $m_b - 1$ at each of these points. We have $|\mathfrak{B}_\sigma| \leq 2g - 2 - \deg(f^*(K_X \otimes S_X)^{\frac{1}{2}})) + |S|$, where g is the genus of B.

Main Theorem (Mok-To (*Crelles* 1991))

Let $\pi : \mathscr{A}_\Gamma \to X_\Gamma$ *be a Kuga family of polarized abelian varieties without locally constant parts,* $\overline{\pi} : \overline{\mathscr{A}_\Gamma} \to \overline{X_\Gamma}$ *be a projective compactification which is a geometic model for the associated modular polarized abelian variety* A_Γ *over* $\mathbb{C}(\overline{X_\Gamma}$. **Then, there are at most a finite number of meromorphic sections of** $\overline{\mathcal{A}_\Gamma}$ **over** $\overline{X_\Gamma}$, *i.e.,* $\mathrm{rank}_{\mathbb{Z}}(A_\Gamma(\mathbb{C}(\overline{X_\Gamma}))) = 0$ *for the Mordell-Weil group* $A_\Gamma(\mathbb{C}(\overline{X_\Gamma})$.

Mordell-Weil Groups Complex Function Fields

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Mordell-Weil group for $f : B \to X_\Gamma$ dominant and equidimensional

Theorem(Mok 1991) *Let* $\Gamma \subset \mathrm{Sp}(g, \mathbb{Z})$ *be torsion-free. Suppose* $\dim(B) = \dim(X_\Gamma)$ *and* $f : B \to X_\Gamma$ *is a dominant classifying map. Denote by* A_f *the elliptic curve over* $\mathbb{C}(\overline{B})$ *obtained by pulling back the universal abelian variety* A_Γ *over* $\mathbb{C}(\overline{X_\Gamma})$ *by the classifying map* f*. Then,*

$$\mathrm{rank}_{\mathbb{Z}} A_f(\mathbb{C}(\overline{B})) \le C \cdot \mathrm{Volume}(R_f, \omega),$$

where ω is the Kähler-Einstein (1,1)-form on X_Γ, C is a universal constant depending only on X_Γ, and R_f is the ramification divisor $f : B \to X_\Gamma$.

The Siegel upper half-plane $\mathcal{H}_g$

$L \subset \mathbb{C}^n$ lattice, $\mathbb{C}/L = A \cong S^1 \times \cdots \times S^1$ (2g copies), $H^1(A, \mathbb{R}) \cong \mathbb{R}^{2g}$ first de Rham cohomology group. **A is called an Abelian variety if $A \hookrightarrow \mathbb{P}^N$ is (projective)-algebraic.**

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Shimura varieties: An example

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The rank-1 case

The complex unit ball $\mathbb{B}^n := \left\{ z \in \mathbb{C}^n : \|z\|^2 < 1 \right\}$

Irreducible Bounded Symmetric Domains

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Classical domains in general

$$D^I(p,q) = \{Z \in M(p,q,\mathbb{C}) : I - \overline{Z}^t Z > 0\}\ , \quad p,q \geq 1$$

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Irreducible Bounded Symmetric Domains

The rank-1 case

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Exceptional domains

D^V, dim 16, **type** E_6; D^{VI}, dim 27, **type** E_7

The André-Oort Conjecture

A point $\tau \in \mathcal{H}$ such that $\tau, j(\tau) \in \overline{\mathbb{Q}}$ is called a **special point** (in which case $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$ by Schneider). The notion of special points is defined for any Shimura variety $X_\Gamma = \Omega/\Gamma$, and the **André-Oort Conjecture** ascertains that the **Zariski closure of any set of special points on X_Γ is a finite union of Shimura subvarieties $X'_{\Gamma'} \hookrightarrow X_\Gamma$.**

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Proof. Assume $\alpha^\beta \in \overline{\mathbb{Q}}$. **Put** $K = \mathbb{Q}(\alpha, \beta, \alpha^\beta)$**;** $f(z) = e^z, g(z) = e^{\beta z}$**;** $x_k = k \log \alpha, k \in \mathbb{N}$ to get a **contradiction**.

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Suppose $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}}$ *are* $\mathbb{Q}$*-linearly independent.* **Then,** $e^{\alpha_1}, \cdots, e^{\alpha_n}$ **are algebraically independent.**

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$$\textbf{trans.deg.}_{\mathbb{Q}}\mathbb{Q}\left(\alpha_1, \cdots, \alpha_n; e^{\alpha_1}, \cdots, e^{\alpha_n}\right) \geq n.$$

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Baker's Theorem (1975)

Suppose $x_1, \cdots, x_n \in \overline{\mathbb{Q}}$, *and* $\log(x_1), \cdots \log(x_n)$ *are linearly independent over* $\mathbb{Q}$. **Then** $1, \log(x_1), \cdots, \log(x_n)$ **are linearly independent over** $\overline{\mathbb{Q}}$

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Closed positive (p,p)-currents (Lelong 1964)

A $\mathscr{C}^\infty$ positive (1,1)-form ω means $\imath \sum \omega_{i\bar{j}}(z) dz^i \wedge d\overline{z^j}$, $\left(\omega_{i\bar{j}}(z)\right) > 0$.

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Techniques from complex geometry

Monotonicity of weighted mass of T over concentric Euclidean balls

Assume T defined on $\mathbb{B}^n(0;R)$. For $0 < r < R$ denote by $m(T;0;r)$ the integral of $T \wedge (\imath\partial\overline{\partial}\|z\|^2)^{n-p}$ over $\mathbb{B}^n(0;r)$; $\nu(T,0;r) := \frac{m(T,0;r)}{\mathrm{Vol}(\mathbb{B}^{n-p}(0;R))}$.

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Recovering complex analytic subvarieties from density conditions

Theorem (Siu [*Invent. Math.* (1970)]) *Let X be a complex manifold, $\dim_{\mathbb{C}}(X) =: n$, $1 \leq p < n$, and T be a closed positive (p,p)-current on X. Let $c > 0$.* **Put** $E_c(T) := \{x \in X : \nu(T;x) \geq c\}$. **Then, $E_c(T) \subset X$ is a complex analytic subvariety where each irreducible subvariety is of complex codimension $\geq p$.**

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Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, $\Gamma \subset \mathrm{Aut}(\Omega)$ be an arithmetic torsion-free lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map.

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Using the above Tsimerman [Ts18] has proven the André-Oort Conjecture for Siegel modular varieties $\mathcal{A}_g = \mathcal{H}_g/\mathrm{Sp}(g;\mathbb{Z})$. Recently, Pila-Shankar-Tsimerman has announced a solution of the full André-Oort Conjecture.

Counting points on definable sets

For a rational point $x = \frac{p}{q}; p, q \in \mathbb{Z}, q \neq 0$, where $|p|$ and $|q|$ are coprime, we define the height $H(x) = \max(|p|, |q|)$. For $x = (x_1, \cdots, x_n) \in \mathbb{Q}^n$) we define $H(x) = \max(H(x_1), \cdots, H(x_n))$.

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Model Theory: o-minimal structures on $\mathbb{R}^n$

A structure $\mathscr{S}$ on $\{\mathbb{R}^n : n \in \mathbb{N}\}$ consists of Boolean algebras of subsets $S_n \subset 2^{\mathbb{R}^n}$ closed under taking Cartesian products and coordinate projections, s.t. $\mathrm{Diag}(\mathbb{R} \times \mathbb{R}) \in S_2$, and, $\mathrm{Graph}(+)$, $\mathrm{Graph}(\times) \in S_3$.

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Theorem (Pila-Wilkie, *Duke J.* 2006)

Let $Z \subset \mathbb{R}^n$ be a definable subset in a given o-minimal structure. Then, $N(Z - Z^{\mathrm{alg}}, T) = T^{o(1)}$, i.e., $\left|Z - Z^{\mathrm{alg}}\right|$ grows subpolynomially.

A generalized Lelong monotonicity formula

Proposition

Let φ *be an unbounded* $\mathscr{C}^\infty$ *strictly psh exhaustion fct on a Stein manifold* X. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be strictly increasing s.t.* $\psi := F \circ \varphi$ *is weakly psh. Let* S *be a closed positive* (p,p)*-current on* X, $0 < p < \dim(X)$. *Then,*

$$h_{S,\varphi}(T) := F'(T)^{n-p} \int_{\{\varphi < T\}} S \wedge (\sqrt{-1}\partial\overline{\partial}\varphi)^{n-p}$$

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Corollary (rough form of Theorem (Hwang-To 2002))

Let (Ω, ds^2_Ω) be a BSD equipped with its Bergman metric, and denote by $\mathbb{B}(x_0; r)$ its geodesic ball of radius r centered at $x_0 \in \Omega$. Let $V \subset \Omega$ be an irr. complex analytic subvariety, $\dim_{\mathbb{C}} V > 0$, passing through x_0. Then, $\exists \lambda = \lambda_\Omega$ and $C = C_\Omega > 0$ such that **Volume** $(\mathbb{B}(x_0; r)) \geq Ce^{\lambda r}$.

Geometric applications of Lelong formulas

1. Lelong's original formula was for closed positive (p, p)-currents on $\mathbb{C}^n$, in which one considers the psh function $\varphi = \|z\|^2$. In this case $\psi := \log \varphi = \log \|z\|^2$ is weakly psh, $F(T) = \log(T)$.

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2. In the case $\mathbb{B}^n$ with potential function $\varphi = -(n+1)\log\left(1 - \|z\|^2\right)$ for the Bergman metric $d(0; z) \sim \varphi(z)$, $\exists c_2 > c_1 > 0$ such that $\{\varphi < c_1 r\} \subset \mathbb{B}(0; r) \subset \{\varphi < c_2 r\}$. Take $F(T) = -e^{-\alpha T}$. We can check that $\exists \alpha > 0$ such that for $\psi := F \circ \varphi = -e^{-\alpha\varphi}$, $\sqrt{-1}\partial\overline{\partial}\psi \geq 0$.

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Ax-Lindemann Theorem for Rank-1 Lattices

Theorem (Mok [Mo19, *Compositio Math.*])

Let $n \geq 2$ *and* $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ *be a not necessarily arithmetic torsion-free lattice. Write* $X_\Gamma := \mathbb{B}^n/\Gamma$, $\pi : \Omega \to X_\Gamma$ *for the uniformization map. Let* $Z \subset \Omega$ *be an irreducible algebraic subset and denote by* $\mathscr{Z} = \overline{\pi(Z)}^{\mathscr{Z}ar} \subset X_\Gamma$ *be the Zariski closure of image of* Z *under the uniformization map in the quasi-projective variety* X_Γ. **Then,** $\mathscr{Z} \subset X_\Gamma$ **is a totally geodesic subset.**

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(a) We have $\mathbb{B}^n \subset \mathbb{P}^n$, Z as an open subset of an algebraic $\widehat{Z} \subset \mathbb{P}^n$ **Consider $[\widehat{Z}]$ as a member of an irreducible component $\mathcal{K}$ of the Chow scheme Chow($\mathbb{P}^n$)**, with associated fiber bundle $\mu : \mathcal{U} \to \mathbb{P}^n$. Restrict $\mathcal{U}$ to $\mathbb{B}^n$ and take quotients wrt Γ to get $\mu_\Gamma : U_\Gamma \to X_\Gamma$. **Prove that $\mathcal{U}_\Gamma$ is algebraic by means of L^2-estimates of $\overline{\partial}$.**

AL Theorem for Rank-1 Lattices (cont.)

(b) Let $\widetilde{\mathscr{Z}}$ be an irreducible component of $\pi_\Gamma^{-1}(\mathscr{Z})$. Then, at a good point $b \in \partial\widetilde{\mathscr{Z}}$, $\widetilde{\mathscr{Z}}$ **extends across b as the union of an analytic family of algebraic subvarieties of $\mathbb{P}^n$**. Let $\mathscr{D}$ be a germ of complex submanifold at b grafted to extend $\widetilde{\mathscr{Z}}$ analytically across b.

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(c) $\mathscr{D} \cap \mathbb{B}^n$ is a local strictly peudoconvex manifold with smooth boundary, and **by Klembeck [Kl87] $\mathscr{D} \cap \mathbb{B}^n$ is asymptotically of constant holomorphic sectional curvature -2, hence asymptotically totally geodesic.**

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(d) By rescaling using elements $\gamma \in \pi_1(\mathscr{Z}) \hookrightarrow \pi_1(X_\Gamma) \cong \Gamma$, it follows that **$\Pi$ is of constant holomorphic sectional curvature -2, hence totally geodesic.**

Compactification Theorem by L^2-estimates of $\overline{\partial}$

Theorem (Mok-Zhong [MZ89, *Ann. Math.*])

Let (X, g) be a complete Kähler manifold. **Assume that** $\mathrm{Vol}(X, g) < \infty$, $\|$**Sectional Curvature**$(X, g)\| < \infty$, *and that X has finite topology.*

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Earlier Ax-Schanuel-type results

Ax-Schanuel Theorem

Theorem (Ax71, *Annals*) *Let $f_1, \cdots, f_n \in \mathbb{C}[[z_1, \cdots, z_m]]$ be $\mathbb{Q}$-linearly independent formal power series with no constant terms.*

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Ax-Schanuel for the j-function

Pila-Tsimerman [PT16] proved an analogue of Ax-Schanuel for the Cartesian product $\mathcal{H}^n$ of upper half-planes, replacing the exponential function by $j : \mathcal{H} \to \mathbb{C}$, thus considering $\mathbb{C}(f_1, \cdots, f_n; j \circ f_1, \cdots, j \circ f_n)$. They also proved an analogue involving at the same time j' and j''.

Ax-Schanuel Theorem on Shimura varieties

Theorem (Mok-Pila-Tsimerman ([MPT19, *Annals*])

Let $\Omega \Subset \mathbb{C}^N$ *be a bounded symmetric domain,* $\Gamma \subset \mathrm{Aut}(\Omega)$ *be an arithmetic lattice, and write* $X_\Gamma := \Omega/\Gamma$*, as a quasi-projective variety. Let* $W \subset \Omega \times X_\Gamma$ *be an algebraic subvariety. Let* $D \subset \Omega \times X_\Gamma$ *be the graph of the uniformization map* $\pi_\Gamma : \Omega \to X_\Gamma$*, and* U **be an irreducible component of** $W \cap D$ **whose dimension is larger than expected**,

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$$\mathrm{codim}\, U < \mathrm{codim}(W) + \mathrm{codim}(D),$$

the codimensions being in $\Omega \times X_\Gamma$, or, equivalently,

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Then, the projection of U to X_Γ is contained in a totally geodesic subvariety $Y \subsetneq X_\Gamma$.

Ax-Schanuel of MPT in terms of functional transcendence

Fix a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Omega), \pi : \Omega \to X_\Gamma$. Modular functions are Γ-invariant meromorphic functions descending to rational functions on X_Γ.

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Let $V \subset \Omega$ be an irreducible complex analytic subvariety, **not contained in any weakly special subvariety $E \subsetneq \Omega$.** *Let $(z_i)_{1 \le i \le n}$ be algebraic coordinates on Ω, $\{\varphi_1, \ldots, \varphi_N\}$ be a basis of modular functions.* **Then,**

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1. We may take the algebraic coordinates $(z_1, \cdots, z_n)$ to be the Harish-Chandra coordinates on $\Omega \Subset \mathbb{C}^n \subset \widehat{\Omega}$.
2. Here a weakly special subvariety $E \subset \Omega$ is a totally geodesic submanifold $E \subset \Omega$ such that $\pi(E) \subset X_\Gamma$ is quasi-projective.

Tame complex geometry

The Definable Remmert-Stein Theorem

Theorem (Peterzil-Starchenko [Proc. ICM 2010]) *Let M be a definable complex manifold and E a definable complex analytic subset of M. Let A be a definable, complex analytic subset of $M - E$.* **Then, its topological closure $\overline{A}$ is a complex analytic subset of M.**

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The Definable Chow Theorem

Theorem (Peterzil-Starchenko, variation of [Proc. ICM 2010]) *Let Y be a quasi-projective algebraic variety. Let $A \subset Y$ be definable, complex analytic, and closed in Y.* **Then, A is algebraic.**

Ax-Schanuel for variations of Hodge structures

Theorem (Bakker-Tsimerman, *Invent. Math.* 2019)

Let X be a nonsingular quasi-projective manifold underlying a **polarized integral variation of Hodge structures**, *$\mathscr{D}$ be the associated* **period domain**, *$\mathscr{D} \subset \check{\mathscr{D}}$ the standard embedding of $\mathscr{D}$ into its dual $\check{\mathscr{D}}$, which is a rational homogeneous manifold.*

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Then, the canonical projection of U to X is contained in **a proper weak Mumford-Tate subvariety**.

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Algebraic subsets of a BSD invariant under cocompact $\check{\Gamma}$

Proposition 1 (Chan-Mok, *JDG* 2021) *Let D and Ω be BSD, $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ be a group homomorphism, $F : D \to \Omega$ be a Φ-equivariant holomorphic map.* **Then, F is totally geodesic.**

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Asymptotic Total Geodesy of Embedded Poincaré Disks

Theorem (Chan-Mok [CM21], *JDG*)

Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding, where* λ *is a positive real constant and* $\Omega \Subset \mathbb{C}^N$ *is a bounded symmetric domain in its Harish-Chandra realization.*

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Embedded Poincaré Disks with $\mathrm{Aut}(\Omega)$-equiv. Tangents

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Proof by rescaling: Compose with $\gamma_i \in \mathrm{Aut}(\Omega)$ **and take limits.**

Total Geodesy of Certain Curves on Tube Domains

Proposition 3

Let Ω be an irreducible bounded symmetric domain of tube type and of rank r; $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent planes spanned by vectors of rank r. **Then, $Z \subset \Omega$ is totally geodesic and of rank r (i.e. of diagonal type).**

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Proof. $\pi : \mathbb{P}T_\Omega \to \Omega$, $L \to \mathbb{P}T_\Omega$ tautological line bundle.
$[\mathscr{S}] \cong L^{-r} \otimes \pi^* E^2$, E dual to $\mathcal{O}(1)$ on the compact dual M of Ω.

$$(2\pi)^{-1}\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 = rc_1(L, \hat{g}_0) - 2c_1(\pi^* E, \pi^* h_0),$$

where $\hat{g}_0$ and h_0 are canonical metrics. $\|s(x)\|$ only depends on the $\mathrm{Aut}(\Omega)$-isomorphism type of $T_x(\Omega)$. Thus, $\|s\|$ = constant on Z.

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$$0 = rc_1(L, \hat{g}_0) - 2c_1(\pi^* E, \pi^* h_0).$$

$\Leftrightarrow$ Gauss curvature $K(x) = -2/r$, and $\sigma \equiv 0$. $\square$

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The original Existential Closedness Problem, raised by Zilber, asks for a minimal set of conditions on an algebraic subvariety of $V \subset \mathbb{C}^n \times (C^*)^n$ to guarantee that $V \cap \mathrm{Graph}(\mathbf{exp})$ is Zariski dense in V. It ties up with the André-Oort and the Zilber-Pink conjectures in Diophantine geometry.

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The proof involves studying the Shilov boundary of $\Omega \Subset \mathbb{C}^N$.

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THANK YOU!!