Geometric parametrization of metric spaces

Chang-Yu Guo

(The Annual Meeting of Complex Analysis of Several Variables)

Shandong University

Shanghai, August 2022

Outline of the presentation

- Background
  1. Nonsmooth analysis and motivations
  2. Geometric parametrization problem
- The quasiconformal Jacobian problem
  1. QC Jacobian problem via Fu's theorem
  2. QC Jacobian problem via QC flows
- Uniformization of metric surfaces
  1. Bonk-Kleiner and Rajala's uniformization
  2. The approach of Lytchak-Wenger
- Two related topics
  1. The Schoen-Li-Wang Conjecture and beyond
  2. Toro's bi-Lipschitz parametrization theorem

Relevant references

Geometric maps in metric spaces:

- Guo-Williams, arXiv, Revised, 2019.
- Guo-Wenger, Comm. Anal. Geom. 2020.
- Guo-Huang-Xu-Wang, Math. Z. (2022, to appear)
- Guo-Hencl-Tengvall, Adv. Cal. Var. 2020.

Related PDEs

- Guo-Xiang, Tran. Amer. Math. Soc. 2021, J. Lond. Math. Soc. 2020.
- Guo-Xiang-Zheng, J. Math. Pures Appl. 2022, Cal. Var. PDEs 2021, arXiv 2022.
- Guo et al., Anal. & PDE (2023, to appear)
- Guo-Xiang-Wang, arXiv 2021.

Part I: Background

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

Singular metric spaces

In the field of "Analysis and geometry on metric (measure) spaces", we regard C^2-Riemannian manifolds as smooth metric spaces and all the other spaces are considered as singular metric spaces.

In particular, the following well-known classes of metric spaces are considered as singular:

- Riemannian manifolds with C^1 or Lipschitz Riemannian metric
- All subRiemannian manifolds and Finsler manifolds
- Alexandrov spaces (with bounded upper/lower curvature)
- Fractional spaces (including graphs, self-similar spaces)
- Weighted Riemannian manifolds or infinite-dimensional metric spaces
- Metric spaces with bounded upper/lower "Ricci" curvature

In a wide sense, research related to analysis and geometry in the framework of singular metric spaces belong to the field of

Analysis and geometry on metric spaces

Sometimes, people also called it nonsmooth analysis or nonsmooth geometry. The initiation of the research on singular metric spaces dates back to the 1990s.

It is worth to point out that in doing Riemannian geometry, one naturally encounters non-smooth spaces

- When taking limits (such as Gromov-Hausdorff) of Riemannian manifolds (procedure used often, e.g. contradiction arguments, blow up arguments, singularities in geometric flows)
- When taking quotients (or cones, or suspensions, or foliations) of Riemannian manifolds
- Mostow's strong rigidity, Margulis super-rigidity, ...

Motivation I: Lipschitz Analysis

Basic question (Early 1990s)

What kind of metric spaces are locally or globally bi-Lipschitz equivalent with the standard Euclidean spaces?

Seminal works:

- David-Semmes (conferences proceedings 1990): Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings
- Semmes (Rev. Mat. Ibero. 1996): On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights

Main motivation: Develop harmonic analysis (such as CZ theory) and geometric measure theory (such as uniform rectifiability) to more general spaces.

Motivation II: Harmonic mappings and metric geometry

Basic question (Middle 1990s)

Extend Riemannian geometry and geometric analysis to singular metric spaces.

Seminal works:

- Gromov-Schoen (Publ. IHES 1992)
- Korevaar-Schoen (Comm. Anal. Geom. 1993)
- Cheeger-Colding (J. Diff. Geom. 1997, 2000)
- Cheeger (GAFA 1999)
- Lott-Villani-Sturm (Ann. Math. 2009, Acta Math. 2006)
- Ambrosio-Gigli-Savaré (Invent. Math. 2014, Duke Math. J. 2014)
- ......

Motivation III: Geometric measure theory

Basic question (Late 1990s)

Extend the geometric measure theory to singular metric spaces. In particular, realize De Giorgi's idea to give a new approach to Plateau's problem (extending the Federer-Fleming theory of currents).

Seminal works:

- Ambrosio-Kirchheim (Acta Math. 2000): Currents in metric spaces
- Ambrosio-Kirchheim (Math. Ann. 2000): Rectifiable sets in metric and Banach spaces

Remark: One crucial motivation was to simplify the FF theory, and to clarify the fact that its main results rely heavily on measure theory and (to a certain extent) not on multilinear algebra and on the theory of forms.

Motivation IV: Geometric function theory

Basic question (Middle 1990s)

Extend the theory of quasiconformal mappings to singular metric spaces. In particular, give a new (and correct) proof of the Margulis-Mostow's theory of quasiconformal mappings in subRiemannian manifolds.

Seminal works

- Heinonen-Koskela (Invent. Math. 1995): Definitions of quasiconformality
- Heinonen-Koskela (Acta. Math. 1998): Quasiconformal maps in metric spaces with controlled geometry

Remark: There is a "small gap" in the original proof of Mostow for his celebrated strong rigidity theorem.

Geometric parametrization problem

Basic question (Geometric parametrizaiton of metric spaces)

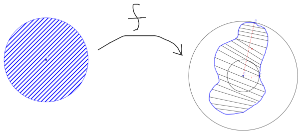
Given a metric space X, when does it admit (locally or globally) a "good geometric parametrization" from the model Euclidean spaces?

Good (Homeomorphic) Geometric Parametrization:

bi-Lipschitz mappings distort the distance up to a bounded multiplicative constant

Quasisymmetric mappings map "round object" to "quasi-round object" (e.g. send balls to ellipsoid)

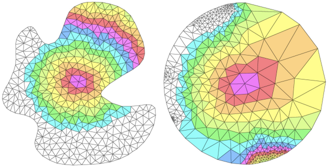
Quasiconformal mappings map "round object" to "quasi-round object" at infinitesimal scale

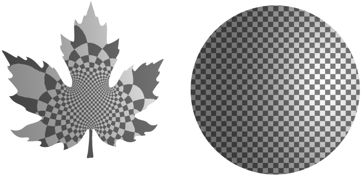


Motivation: uniformization theorem

Example 1 (Riemann uniformization theorem)

Each simply connected Riemann surface is conformally equivalent to one of the three model spaces: the unit disk $\mathbb{D}^2$, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.





Uniformization in higher dimensions

Liouville breaks our dreams to find an analogy of this result in higher dimensions: *conformal mappings are rigid in high dimensions*.

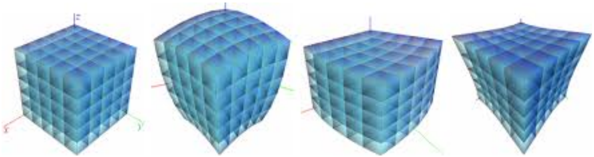


Figure from Google Images

In order to find similar results in higher dimensions, we have to enlarge the class of mappings that are allowed for classification.

$$\Big\{\text{Conformal mappings}\Big\} \subsetneq \Big\{\text{Nice Geometric mappings}\Big\}$$

Uniformization in higher dimensions

Liouville breaks our dreams to find an analogy of this result in higher dimensions: *conformal mappings are rigid in high dimensions*.

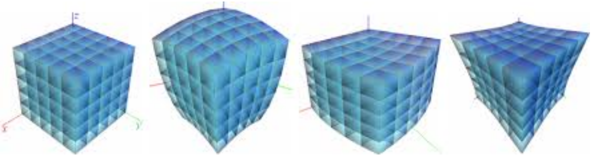


Figure from Google Images

In order to find similar results in higher dimensions, we have to enlarge the class of mappings that are allowed for classification.

$$\Big\{\text{Conformal mappings}\Big\} \subsetneq \Big\{\text{Nice Geometric mappings}\Big\}$$

Motivation: geometric function theory

Geometric function theory is a field where we study geometry of mappings, that is, mappings or deformations between subsets of the Euclidean spaces $\mathbb{R}^n$, and more generally between manifolds or other geometric objects.

Moral of geometric function theory: in order to understand (the geometry of) the space, one can alternatively understand its mappings.

$$\text{Euclidean space} \overset{\text{a good geometric mapping } f}{\curvearrowright} \text{a metric space } X$$

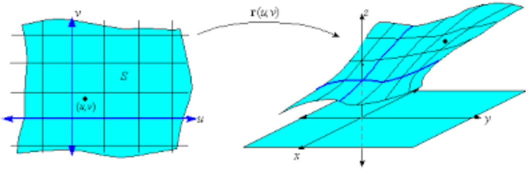


Figure from Google Images

Motivation: geometric function theory

Geometric function theory is a field where we study geometry of mappings, that is, mappings or deformations between subsets of the Euclidean spaces $\mathbb{R}^n$, and more generally between manifolds or other geometric objects.

Moral of geometric function theory: in order to understand (the geometry of) the space, one can alternatively understand its mappings.

$$\text{Euclidean space} \overset{\text{a good geometric mapping } f}{\curvearrowright} \text{a metric space } X$$

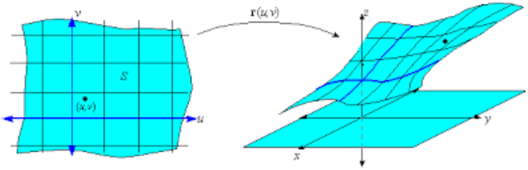


Figure from Google Images

Successfully applications of quasiconformal mappings I

PDE Gehring's improved regularity and higher integrabilty.

Complex dynamics Astala's quasiconformal area distortion estimates.

Inverse problems Astala-Päivärinta's solution for the Calderon's inverse conductivity problem.

Differential geometry Mostow's celebrated strong rigidity of locally symmetric spaces. (uniqueness of hyperbolic structures for $n \geq 3$).

Geometric topology Sullivan's uniformisation theorem - the existence of quasiconformal structures on topological n-manifolds.

Tukia-Väisälä's "quasiconformal geometric topology".

Connes-Sullivan-Teleman's theory of characteristic classes on topological manifolds.

Successfully applications of quasiconformal mappings II

Mathematicla Physics Donaldson–Sullivan's "quasiconformal Yang-Mills theory".

Analysis on metric spaces Heinonen-Koskela's theory of quasiconformal mappings in metric spaces with controlled geometry.

The above is an in-complelte list of the very successful applications of quasiconformal mappings. Many other important applications (such as Nonlinear potential theory, nonlinear elasticity, quasiconformal group action, geometric group theory, ...) should also be mentioned.

Conclusion: A general theory of quasiconformal mappings beyond Riemannian manifolds is necessary and remarkable.

Part II: Quasiconformal Jacobian problem

The quasiconformal Jacobian problem

Quasiconformal Jacobian problem (David-Semmes, 1990)

Which nonnegative functions are comparable to the Jacobian determinant $J_f = \det(Df)$ of a quasiconformal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$?

- Due to Burago-Kleiner (GAFA 98) and McMullen (GAFA 98), there exists a weight ω on $\mathbb{R}^n$ such that both ω and ω^{-1} belong to $L^\infty(\mathbb{R}^n)$, but there is no bi-Lipschitz mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $J_f(x) = \omega(x)$ a.e. on $\mathbb{R}^n$.
- When $n = 2$, the quasiconformal Jacobian problem is equivalent to the well-known bi-Lipschitz parametrization problem: which metric spaces are bi-Lipschitz equivalent to $\mathbb{R}^n$, $n \geq 2$?

An interesting application of the QC Jacobian problem

Global bi-Lipschitz parametrization of conformally flat Riemannian manifolds (Bonk–Heinonen–Saksman, Duke Math. J. 2008).

Observation (Bonk–Heinonen–Saksman, Duke Math. J. 2008)

Consider a conformal deformation g of $\mathbb{R}^n$, i.e. $g = e^{2u} g_0$ for some smooth $u\colon \mathbb{R}^n \to \mathbb{R}$. Then

$$X = (\mathbb{R}^n, g) \overset{\text{bi-Lipschitz}}{\simeq} (\mathbb{R}^n, g_0),$$

if and only if the weight $w = e^{nu}$ is comparable to a quasiconformal Jacobian.

Some basic facts about quasiconformal Jacobian

- By a deep result of Gehring (Acta Math. 1973), the Jacobian of a quasiconformal map is an A_∞ weight, i.e., there exist $\varepsilon > 0$ and $C \geq 1$ such that

$$\left(\int_B \omega^{1+\varepsilon}(x)dx \right)^{1/(1+\varepsilon)} \leq C \int_B \omega(x)dx \qquad (1)$$

  for each open ball $B \subset \mathbb{R}^n$.

- Due to Reimann (CMH 1974), $\log(J_f) \in BMO(\mathbb{R}^n)$, where $BMO(\mathbb{R}^n)$ consists of all locally integrable functions u in $\mathbb{R}^n$ such that

$$\sup_B \int_B |u(x) - u_B| dx < \infty.$$

  Moreover, quasiconformal maps preserve BMO functions (and indeed also A_∞ measures).

An A_1 weight fails to be QC Jacobian

Example (Bishop, Contem. Math. 2007)

There is an A_1 weight ω on $\mathbb{R}^2$ which is not comparable to any quasiconformal Jacobian.

Recall that a locally integrable nonnegative function ω is called an A_1 weight if there is a $C < \infty$ so that

$$\fint_B \omega \leq C \operatorname{essinf}_B \omega$$

for each open ball $B \subset \mathbb{R}^2$.

Open question: are A_1 weights preserved by quasiconformal maps?

Strong A_∞ weights

In order to investigate the QC Jacobian problem, David and Semmes introduced the class of strong A_∞ weight and studied its connection with functional analysis, harmonic analysis and geometric measure theory, leading to the so-called strong A_∞ geometry.

Theorem (Bonk-Heinonen-Saksman, Contem. Math. 2004)

Let $n \geq 2$, $0 < \alpha < n$ and let u belong to the Bessel potential space $L^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$. Then $\omega = e^{nu}$ is a strong A_∞ weight with data depending only on n, α and the $L^{\alpha,\frac{n}{\alpha}}$-norm of u.

Open question (BHS, Contem. Math. 04): is ω as in the above theorem actually comparable to a quasiconformal Jacobian?

With C.-L. Xiang, we made some small progress in the easier case $\alpha = \frac{n}{2}$ (which is related to the operator $(-\Delta)^{\frac{n}{2}}$).

A positive result in dimension two

Theorem (Bonk-Heinonen-Saksman, Contem. Math. 2004)

Let u be a locally integrable function in $\mathbb{R}^2$ with distributional gradient $\nabla u \in L^2(\mathbb{R}^2)$. Then e^{2u} is comparable to a quasiconformal Jacobian.

- The statement is quantitative in the sense that the distortion of the quasiconformal map and the comparability constant depends only on the L^2-norm of ∇u.
- Even in dimension $n = 2$, when $0 < \alpha < 1$, the Open Question remains unsolved.

Key ingredient: a theorem of Fu

Theorem (Fu, Indiana Univ. Math. J. 1998)

Let X be a complete Riemannian 2-manifold that is homeomorphic to $\mathbb{R}^2$. There are absolute constants $\varepsilon_0 > 0$ and $L_0 > 0$ with the following property: if the integral (Gauss) curvature of X is less than ε_0, then X is bi-Lipschitz equivalent to $\mathbb{R}^2$ with bi-Lipschitz constant L_0.

- Bonk and Lang (Math. Ann. 2003) proved that $\varepsilon_0 = 2\pi$ is the optimal bound.

Lemma (approximate harmonic decomposition)

Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. For each $\varepsilon > 0$, there exists a decomposition $u = s + b$ into two compactly supported smooth functions such that $\|\Delta s\|_1 \leq \varepsilon$ and

$$\|b\|_\infty \leq \frac{C}{\varepsilon}\|\nabla u\|_2^2.$$

Proof of BHS's theorem in dimension 2

- Consider the Riemannian manifold $X_u = (\mathbb{R}^n, e^{2u}g_0)$
- For any $b \in L^\infty$, X_{u-b} and X_u are bi-Lipschitz equivalent with $L = e^{\|b\|_\infty}$
- Decompose $u = s + b$ with $\|\Delta s\|_1 < \varepsilon_0$
- The Gauss curvature of $X_s(= X_{u-b}) = (\mathbb{R}^n, e^{2s}g_0)$ is

$$K_s = -e^{-2s}\Delta s$$

  and so

$$\int_{X_s} |K_s| dV_s = \int_{\mathbb{R}^n} |\Delta s| dx < \varepsilon_0.$$

- By Fu's theorem, X_s is bi-Lipschitz equivalent to $\mathbb{R}^n$ and so is X_u.

Paneitz operator and Q-curvature

In dimension 4, there is a Paneitz operator $P = \Delta^2 +$ lower order term.

- On $(\mathbb{R}^4, g_0)$, $P = \Delta^2$
- If $g_1 = e^{2u} g_0$, then

$$P_0 u = Q_1 e^{4u} - Q_0 = Q_1 e^{4u}$$

and so $\|P_0 u\|_1 = \|Q_1 e^{4u}\|_1$.

Above, Q is the Q-curvature from conformal geometry.

However, to mimic the 2d proof in 4d, we need

- a version of Fu's theorem in higher dimensions
- decompose u as $u = s + b$, with $\|b\|_\infty < 0$ and

$$\|\Delta^2 s\|_1 < \varepsilon_0.$$

Logarithmic potentials

A function $u\colon \mathbb{R}^n \to [-\infty, \infty]$ is said to be a logarithmic potential if u is finite almost everywhere and if there is a signed Radon measure μ of finite total variation on $\mathbb{R}^n$ such that

$$u(x) = L\mu(x) := -\int_{\mathbb{R}^n} \log|x-y|d\mu(y)$$

for almost every $x \in \mathbb{R}^n$.

- A potential $L\mu$ is finite almost everywhere if and only if

$$\int_{\mathbb{R}^n} \log^+ |y|d|\mu|(y) < \infty.$$

- $L\mu \in BMO(\mathbb{R}^n)$ and thus it lies in $L^p_{loc}(\mathbb{R}^n)$ for all $p \geq 1$.

QC Jacobian via logarithmic potentials

Theorem (Bonk-Heinonen-Saksman, Duke Math. J. 2008)

For each $n \geq 2$, there exists a positive constant c_n with the following property. If μ is a signed measure on $\mathbb{R}^n$ so that $L\mu$ is finite a.e. and $\|\mu\| < c_n$, then the weight $\omega = e^{nu}$ for the logarithmic potential $u = L\mu$ is comparable to a quasiconformal Jacobian.

Two interesting challenging open problems:

- BHS conjecture that $c_n = 1$ for all $n \geq 2$; when $n = 2$, it is known due to Bonk-Lang (Math. Ann. 2003)
- BHS asks whether Theorem remains valid if we only assume $\|\mu^+\| < c_n$ and $\|\mu^-\| < \infty$?

Q-curvature

The Green's function of Δ in dimension 2 and Δ^2 in dimension 4 is

$$G(x,y) = c_n \log \frac{1}{|x-y|}.$$

So in dimension 2 and 4, if $g = e^{2u}|dx|^2$, then

$$u(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q(y) e^{nu(y)} dy + C, \tag{2}$$

where in dimension 2, Q is the Gaussian curvature and in dimension 4, Q is the so-called Q-curvature.

In dimension 4, if M is CFM, then (4d Gauss-Bonnet-Chern formula)

$$4\pi^2 \chi(M) = \int_M Q.$$

Fu's theorem in higher dimensions

Theorem (Bonk-Heinonen-Saksman, Duke Math. J. 2008)

There exists an $\varepsilon > 0$ such that if $g = e^{2u} g_0$ is a smooth normal metric on $\mathbb{R}^n$ for n even and if

$$\int_{\mathbb{R}^n} |Q| < \varepsilon,$$

then $(\mathbb{R}^n, e^{2u} g_0)$ is bi-Lipschitz equivalent to $(\mathbb{R}^n, g_0)$.

- The result holds for all even n and when $n = 2$, Q is the scalar/Gaussian curvature
- Wang (IMRN 2012) improved this result for an optimal bound on $\int Q^+$
- Chang-Prywes-Yang (Adv. Math. 2022) extended this result to the model space $\mathbb{S}^4$ with any positive Yamabe metric g

Nontriviality of the bi-Lipschitz parametrization

In dimension 2

- If $(M, g) = (\mathbb{R}^2, g)$, then for some $\|\mu\|_\infty < 1$,

$$g = a(z)\Big|dz + \mu(z)d\bar{z}\Big|.$$

- Solving the Beltrami equation for μ, we conclude that every metric g is conformally equivalent to $(\mathbb{C}, |dz|^2)$

However, without further assumption on g (such as integral of the Gauss curvature is small), the metric g is not necessarily bi-Lipschitz equivalent to our model space.

- Due to Siebenmann-Sullivan, for each integer $n \geq 5$, there are finite n-dimensional polyhedra that are topological manifolds but not locally bi-Lipschitz equivalent to the ball in $\mathbb{R}^n$ (w.r.t. the natural intrinsic metric in the polyhedron).

Connection with isoperimetric inequalities

Theorem (Wang, Adv. Math. 2015)

If $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ is a complete noncompact even dimensional manifold. Let Q^+ and Q^- denote the positive and negative part of Q_g. Suppose $g = e^{2u}|dx|^2$ is normal, i.e.,

$$u(x) = c_n^{-1} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) dv_g(y) + C.$$

Then if the Q-curvature satisfies

$$\alpha := c_n^{-1} \int_M Q^+ dv_g < 1$$

and

$$\beta := c_n^{-1} \int_M Q^- dv_g < \infty,$$

then (M, g) satisfies the isoperimetric inequality with isoperimetric constant depending only on n, α and β.

QC flow of Reimann

Let $V\colon \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ be a time-dependent vector field in $\mathbb{R}^n$ and consider the ODE

$$\frac{df}{dt}(x,t) = V(f(x,t),t), \quad f(x,0) = x.$$

Theorem (Reimann, Invent. Math. 1976)

If V is continuous and $W^{1,1}_{loc}$ and $\mathcal{S}V = \frac{1}{2}(DV + DV^T) - \frac{1}{n}\text{div}(V)I_{n\times n}$ is in L^∞, then f exists and is e^{2ct}-quasiconformal with $c = \|\mathcal{S}V\|_\infty$.

- $\mathcal{S}V$ was first introduced by Ahlfors to measure quasiconformal deformations and is called the Ahlfors conformal strain tensor
- In dimension $n = 2$, $\mathcal{S}V = \partial_{\bar{z}}V$

The Jacobian formula for flows

Key proposition (Bonk-Heinonen-Saksman, Duke Math. J. 2008)

If f is the flow map of a continuous Sobolev vector field V with $\|\mathcal{S}V\|_\infty < \infty$, then

$$\log J_{f_t} = \int_0^t \mathrm{div}(V)(f_s(x), s)ds,$$

where J_{f_t} is the Jacobian determinant of Df_t.

- If $g = e^{2u}g_0$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-QC with $J_f \approx e^{2u}$, then f is (quantitatively) bi-Lipschitz from $(\mathbb{R}^n, g)$ to $(\mathbb{R}^n, g_0)$
- On $\mathbb{R}^4$, $\Delta^2 u = Qe^{4u}$, so if Δ^2 is invertible by an operator G, then u can be expressed as $G(Qe^{4u})$
- So the strategy is to construct a vect. field V with $\|\mathcal{S}V\|_\infty < \infty$ and $\mathrm{div}(V) \approx G(Qe^{4u})$.

Construction of the vector field V

Let $v(x,y) = -\log|x-y|(x-y)$. Then

- $\mathrm{div}(v(x,y)) \approx -n\log|x-y|$
- $|\mathcal{S}v|$ is uniformly bounded

The essential idea is that v corresponds to the logarithm potential of the Dirac measure δ in the sense that

$$L\delta = \frac{1}{n}\mathrm{div}(v) + b + c,$$

where b is L^∞ and c is constant.

For a general logarithm potential of the form $L\varphi$ with $\varphi = Qe^{nu}$, it suffices to consider

$$V(x) := \int_{\mathbb{R}^n} v(x,y)\varphi(y)dy.$$

BLD parametrization

Theorem (Heinonen-Sullivan, Duke Math. J. 02)

Under some natural topological constrains on X, X is locally branched Euclidean if and only if it supports a Poincaré inequality and the Cartan-Whitney presentations exist locally on X.

- Here locally branched Euclidean refers to admit a local BLD-maps into $\mathbb{R}^n$. Roughly speaking, BLD maps are branched version of bi-Lipschitz map in the sense that they quasi-preserve length of (nonconstant) curves.
- Under higher Sobolev regularity for the Cartan-Whitney presentation, Heinonen and Keith (Publ. IHES 2011) proved the existence of local bi-Lipschitz parametrization.
  Topological $+$ analytical $\rightarrow$ bi-Lipschitz parametrization

More on BLD Euclidean spaces

Heinonen-Rickman (Duke Math. J. 02)

- A locally BLD-Euclidean space is a Lipschitz manifold outside a close singular set of topological dimension at most $n-2$
- Open Question: Whether locally BLD-Euclidean spaces are bi-Lipschitz embeddable in some finite-dimensional Euclidean spaces?
- Conjecture: a locally BLD-Euclidean space X is locally metrically orientable.
- Open Question: Whether the branch set of an L-BLD map has Hausdorff dimension strictly less than the dimension of the space?

Guo and Willliams [arXiv 2019] proved that

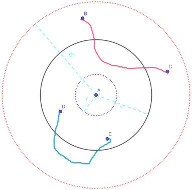
- The answer to the two open questions are affirmative, but, the conjecture is false.

Part III: Uniformization of metric surfaces

Necessary conditions for bi-Lipschitz parametrization

If X is bi-Lipschitz equivalent to $\overline{\mathbb{D}}$ or $\mathbb{S}^2$, then

- X is Ahlfors 2-regular: $C^{-1}r^2 \leq \mathcal{H}^2\left(B(x,r)\right) \leq Cr^2$;
- X is linearly locally connected (LLC).



The LLC condition restricts geometry: no cusps, no neck-pinch.

Example 2 (Laakso, Geom. Funct. Anal. 02')

There exists a metric space X, Ahlfors 2-regular, LLC, homeomorphic to $\mathbb{D}$ such that X does not bi-Lipschitz embed into any Hilbert space.

Quasisymmetric uniformization of metric spheres

Theorem (Bonk-Kleiner, Invent. Math. 02')

A metric space homeomorphic to $\mathbb{S}^2$ which is Ahlfors 2-regular and LLC is quasisymmetric to $\mathbb{S}^2$.

- The Bonk-Kleiner theorem has important application in Cannon's conjecture in geometric group theory (see Bonk's ICM talk and Kleiner's ICM talk)
- There are many other uniformization results for fractal spaces, especially on Sierpiński type spaces: Bonk (Amer. J. Math. 09, Invent. Math. 11, Ann. Math. 13), Merenkov (Invent. Math. 10), Ntalampekos-Younsi (Invent. Math. 2020)

Quasiconformal uniformization of metric surfaces

Theorem (Rajala, Invent. Math. 17')

There exists a quasiconformal mapping $f\colon X \to \Omega \subset \mathbb{R}^2$ if and only if X is reciprocal.

- Rajala's Theorem $\Longrightarrow$ Bonk-Kleiner's Theorem;
- The essential idea of Rajala relies on some old techniques of Gehring and Vaisala, which essentially uses the close connection between planar analytic function and (real-valued) harmonic function.

The reciprocal condition of Rajala

Definition (Reciprocal)

A metric space X is called reciprocal if for every $x \in X$ and every $R > 0$ with $X \backslash B(x, R) \neq \emptyset$ we have

$$\lim_{r \to 0} \mathrm{Mod}(B(x,r), X \backslash B(x,R), \overline{B(x,R)}) = 0 \tag{3}$$

and there exists $\kappa > 0$ such that every closed topological square $Q \subset X$ with boundary edges $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order satisfies

$$\mathrm{Mod}(\xi_1, \xi_3, Q) \cdot \mathrm{Mod}(\xi_2, \xi_4, Q) \leq \kappa. \tag{4}$$

- One always has

$$\mathrm{Mod}(\xi_1, \xi_3, Q) \cdot \mathrm{Mod}(\xi_2, \xi_4, Q) \geq \kappa^{-1}. \tag{5}$$

A new approach via parametrized Plateau's problem

The parametrized Plateau problem

Given a Jordan curve Γ in a metric space X, is there a disc-type surface that minimizes the area of all such surfaces with boundary Γ?

Figure from Google Images

Solutions to Plateau's problem in metric spaces

Theorem (Lytchak-Wenger, ARMA 2017)

The parametrized Plateau problem is solvable in all proper metric spaces.

Some further extensions:

- Guo and Wenger (Comm. Anal. Geom. 2020) solved the Plateau problem also in certain locally non-compact metric spaces including all dual Banach spaces and Hadamard spaces
- Fitzi and Wenger (J. Reine Angew. Math. 2021) considered area minimizers with bounded genus
- More recently, Creutz-Fitzi (arXiv 2021) considered area minimizers with bounded genus for singular curve $\Gamma \subset X$

Geometric parametrization via Plateau's problem

Connection: we may regard solutions u of the Plateau problem as good geometric parametrization, provided that u is topologically nice (e.g. injectivity) and analytically nice (e.g. regularity)

$$\left\{\text{Regularity of area minimizing}\right\} \leftrightsquigarrow \left\{\text{Regularity of energy minimizing}\right\}$$
$$\leftrightsquigarrow \left\{\text{Regularity of quasiconformal mapping}\right\}$$

Topological property of $u \rightsquigarrow$ Bonk-Kleiner's quasisymmetric uniformization theorem (2002), Rajala's quasiconformal uniformization theorem (2017)

Analytic property of $u \rightsquigarrow$ regularity of harmonic mappings in singular metric setting

Geometric parametrization via Plateau's problem

Connection: we may regard solutions u of the Plateau problem as good geometric parametrization, provided that u is topologically nice (e.g. injectivity) and analytically nice (e.g. regularity)

$$\Big\{\text{Regularity of area minimizing}\Big\} \leftrightsquigarrow \Big\{\text{Regularity of energy minimizing}\Big\}$$
$$\leftrightsquigarrow \Big\{\text{Regularity of quasiconformal mapping}\Big\}$$

Topological property of $u \rightsquigarrow$ Bonk-Kleiner's quasisymmetric uniformization theorem (2002), Rajala's quasiconformal uniformization theorem (2017)

Analytic property of $u \rightsquigarrow$ regularity of harmonic mappings in singular metric setting

Geometric parametrization via Plateau's problem

Connection: we may regard solutions u of the Plateau problem as good geometric parametrization, provided that u is topologically nice (e.g. injectivity) and analytically nice (e.g. regularity)

$$\Big\{\text{Regularity of area minimizing}\Big\} \leftrightsquigarrow \Big\{\text{Regularity of energy minimizing}\Big\}$$
$$\leftrightsquigarrow \Big\{\text{Regularity of quasiconformal mapping}\Big\}$$

Topological property of $u \rightsquigarrow$ Bonk-Kleiner's quasisymmetric uniformization theorem (2002), Rajala's quasiconformal uniformization theorem (2017)

Analytic property of $u \rightsquigarrow$ regularity of harmonic mappings in singular metric setting

Bonk-Kleiner's QS uniformization via Plateau's problem

Theorem (Lytchak-Wenger, Duke Math. J. 2020)

Let X be a geodesic metric space homeomorphic to $\overline{\mathbb{D}}$ and such that $l(\partial X) < \infty$. If X is Ahlfors 2-regular and LLC, then X is quasisymmetric to $\overline{\mathbb{D}}$.

Main steps:

- There exists $u \in \Lambda(\partial X)$ such that u is energy minimizing in $\Lambda(\partial X)$ and hence is $\sqrt{2}$-qc and u is continuous in $\overline{\mathbb{D}}$;
- Show that u is injective and thus it is a quasiconformal homeomorphism;
- The method of Heinonen-Koskela (Acta Math. 98) yield upgrade to quasisymmetric.

Topological consequence for energy minimizers

Key proposition (Lytchak-Wenger, Duke Math. J. 2020)

Let X be a geodesic metric space homeomorphic to $\mathbb{D}$, and let $u\colon \overline{\mathbb{D}} \to X$ be a continuous map. If u is an energy minimizer among all possible candidates for the Plateau problem, then u is monotone.

Recall that a continuous map $u\colon \mathbb{D} \to X$ is monotone if $u^{-1}(x)$ is a connected set for each $x \in X$.

When X is a topological disk, u is monotone if and only if it is the uniform limit of homeomorphisms $u_n\colon \mathbb{D} \to X$.

Rajala's QC uniformization via Plateau's problem

Theorem (Meier-Wenger, preprint 2021)

Let X be a locally geodesic metric space homeomorphic to $\mathbb{R}^2$ and of locally finite Hausdorff 2-measure. If $\Omega \subset X$ is a Jordan domain of finite boundary length, then there exists a continuous, monotone surjection $u\colon \mathbb{D} \to \Omega$ such that

$$\mathrm{Mod}(\Gamma) \leq K\mathrm{Mod}(u \circ \Gamma). \tag{6}$$

- Meier-Wenger Theorem $\Longrightarrow$ Rajala's QC uniformization theorem
- If $u\colon \mathbb{D} \to \Omega$ is a homeomorphism with (6), then it is called a geometric quasiconformal map and it is equivalent to the analytic definition
- There is a further new proof by Ntalampekos-Romney (Duke Math. J. 2022) via polyhedral approximation of metric surfaces.

Key ingredient of Meier-Wenger

Key ingredient (Meier-Wenger, 2021)

Let X be a locally geodesic metric space homeomorphic to $\mathbb{D}$ and let $\Gamma \subset X$ be a Jordan curve. Let $u\colon \mathbb{D} \to X$ be a energy minimizing candidate. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathrm{osc}(u, z, \delta) < \varepsilon$ for every $z \in \mathbb{D}$.

Recall that $\mathrm{osc}(u, z, \delta) := \inf\{\mathrm{diam}(u(E)) : E \subset \mathbb{D} \cap B(z, \delta)$ subsets of full measure $\}$

- Key ingredient implies both interior and boundary continuity
- Together with Key proposition implies u is monotone
- Regularity of energy minimizer gives u is infinitesimally quasiconformal
- Equivalence of analytic and geometric quasiconformality $\Longrightarrow$ Rajala's QC uniformization

Koebe's conjecture and progress

Recall the well-known Koebe conjecture predicts that every domain $\mathbb{D} \subset \hat{\mathbb{C}}$ admits a conformal map onto a circle domain, i.e. a domain whose set of complementary components consists of closed disks and points.

A major breakthrough was achieved by He and Schramm.

Theorem (He-Schramm, Ann. Math. 1993)

Keobe's conjecture holds for countably connected domains.

- He-Schramm (Invent. Math. 1994) also made progress on certain uncountably connected domains
- In a recent breakthrough, Rajala [arXiv 2021] extends (and reproves) the He-Schramm theorem by exhaustion technique (approximation from inside)

Part IV: Two related topics

The Schoen-Li-Wang Conjecture

The Schoen-Li-Wang Conjecture

Every quasiconformal self-homeomorphism of the boundary at infinity of a rank one symmetric space M extends to a unique harmonic map from M to itself.

This conjecture has recently been solved in the affirmative in a series of break-through papers by

- Markovic (Invent. Math. 2015, J. Amer. Math. Soc. 2017)
- Lemma-Markovic (J. Diff. Geom. 2018)
- Benoist-Hulin (Ann of Math. 2017)

Benoist-Hulin's theorem: beyond the SLC conjecture

Theorme (Benoist-Hulin, Ann. Math. 2017)

Every quasi-isometric map between rank one symmetric spaces X and Y is at finite distance of a unique harmonic map.

- The Benoist-Hulin theorem $\Longrightarrow$ Schoen-Li-Wang Conjecture
- Benoist-Hulin (J. Euro. Math. Soc. 2020) further extended this result to the case when X and Y are Hadamard manifolds of pinched negative curvature, i.e. simply connected Riemannian manifolds of sectional curvature bounded by $-b^2 \leq K_X, K_Y \leq -a^2 < 0$

Sidler-Wenger's extension

Theorem (Sidler-Wenger, J. Diff. Geom. 2021)

Let X be a Hadamard manifolds of pinched negative curvature and let (Y, d_Y) be a proper Gromov hyperbolic $CAT(0)$-space. Then for every quasi-isometric map $f\colon X \to Y$, there exists an energy minimizing harmonic map $u\colon X \to Y$ which is globally Lipschitz continuous and

$$\sup_{x\in X} d_Y\big(u(x), f(x)\big) < \infty.$$

Guo and Zhang proved that

- The properness assumption in Sidler-Wenger's result can be removed
- Key ingredient: show the compactness of minimizing harmonic maps between certain Alexandrov spaces and uses a Rellich compactness theorem for locally noncompact spaces.

Theorem (Toro, J. Diff. Geom. 1994)

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $u \in W^{2,2}(\Omega, \mathbb{R})$. Denote by $\Gamma_u = \{(x, u(x)) : x \in \overline{\Omega}\}$ the graph of u. Then Γ_u is homeomorphic to a subdomain Ω' of $\mathbb{R}^2$ through a bi-Lipshitz homeomorphism. More precisely, there exists a homeomorphism $\Phi\colon \Omega' \to \Gamma_u$ and $L > 0$ such that

$$\begin{cases} |\Phi(z) - \Phi(z')| \le L|z - z'| & \text{for all } z, z' \in \Omega', \\ |\Phi(Z) - \Phi(Z')| \le L|Z - Z'| & \text{for all } Z, Z' \in \Gamma_u, \end{cases}$$

and furthermore,

$$L \le C\left(1 + \|u\|_{W^{2,2}(\Omega)}\right).$$

Nontriviability of Toro's theorem

Example 3 (A simple but typical example)

Let $D \subset \mathbb{R}^2$ be a disk of radius $1/2$ and define $u\colon D \to \mathbb{R}$ by

$$u(x,y) = x \log|\log r|, \quad r = \sqrt{x^2+y^2}.$$

Direct computation gives $|D^2u| \lesssim r^{-1}|\log r|^{-1}$ and so $D^2u \in L^2(D)$. On the other hand,

$$u_x = \log|\log r| + O(|\log r|^{-1}) \quad \text{and} \quad u_y = O(|\log r|^{-1})$$

as $r \to 0$. This suggests that Du is unbounded around the origin.

The unit normal of Γ_u, the graph of u, is

$$\nu = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}} \to -e_1 = (-1,0,0)$$

as $r \to 0$ and hence Γ_u is a C^1 surface in $\mathbb{R}^3$.

Theorem (Toro, Duke Math. J. 95)

There exists $\varepsilon > 0$, such that every surface S in $\mathbb{R}^n$ with the following conditions, is Lipschitz:

- There exist a sequence of smooth surfaces S_k in $\mathbb{R}^n$, which converges in measure in $\mathbb{R}^n$ to S
- $\forall x \in \mathbb{R}^n$, there exist a ball $B(x, \rho)$ and $\beta > 0$ such that

$$\mathcal{H}^2(S_k \cap B(x, \rho)) \leq \beta$$

  and

$$\int_{S_k \cap B(x,\rho)} |A_k|^2 d\mathcal{H}^2 \leq \varepsilon^2,$$

  where A_k is the second fundamental form of S_k embedded in $\mathbb{R}^n$.

Remark: This is connected to the Willmore surface problem.

An approach due to Muller-Sverak

Theorem (Muller-Sverak, J. Diff. Geom. 95)

Conformal parametrizations of surfaces with square-integrable second fundamental form are locally bi-Lipschitz homeomorphisms.

- This approach relies on the compensation by compactness phenomena
- One useful observation is the Coulomb orthonormal moving frames on a surface is closed related to the conformal coordinates on this surface
- Surfaces with mean curvature in L^2 are of general interest in geometric measure theory. There is a well-known Lamm-Sharp Conjecture (Comm. PDEs. 2016) claiming weakly conformal solution of $-\Delta u = \Omega \cdot \nabla u$ will be locally in $W^{2,2} \cap W^{1,\infty}$.

If you fall in love with harmonic functions, your mathematician's soul will never come to rest unless you comprehend the origin of their irresistible appeal and beauty. And if you are bent on spaces, manifolds and maps you start researching for the geometric habitat of harmonicity.

MIKHAIL GROMOV