Finite type conditions for real smooth hypersurfaces

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1. Cartan-Chern-Moser Theorem
2. Kohn's Sub-Elliptic Estimates

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Cartan: The strongly pseudoconvex real analytic hypersurface near the origin in $\mathbb{C}^2$ possesses the following normal form:

$$v = |z|^2 + \sum_{k,l \geq 2, k+l \geq 6} a_{k\bar{l}}(u) z^k \overline{z}^l.$$

Here $(z, w = u + iv)$ are the coordinates of $\mathbb{C}^2$.

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Chern-Moser: Normal form for real hypersurfaces in $\mathbb{C}^n$.

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A natural question: What's the local holomorphic invariants for general pseudoconvex hypersurfaces?

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A natural question: Do the sub-elliptic estimates hold for general pseudoconvex domains?

Kohn showed that the sub-elliptic estimates does not always hold for general pseudoconvex domains.
If D is a domain defined by

$$\left\{r < 0,\ r(z_1, z_2, w) = Re(w) + |z_1^2 + z_2^3|^2 + exp^{-(|z_1|^2+|z_2|^2+|w|^2)^{-1}}.\right\}$$

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A natural question: What kind of pseudoconvex domains possess sub-elliptic estimates ?

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L: a $(1,0)$ tangential vector field near p with $L(p) \neq 0$.

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Here and in what follows, ρ is the defining function of M near p.

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When z is required to be regular, this is exactly the regular finite type $a^{(1)}(M,p)$.

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This finite type at $0 \in M$ is of finite type m if and only if the defining function can take the following form

$$\rho = \text{Im}(w) + P(z,\overline{z}) + O(|z|^{m+1} + |z\text{Re}(w)| + |\text{Re}(w)|^2).$$

Here P is a non trivial homogeneous polynomial of degree m without harmonic terms.

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3. The s-type of the Levi form $c^{(s)}(M,p)$.

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- The third invariant is defined by the degeneracy of the Levi form, it is always more easily to be applied.

- **Bloom-Graham (1977):** $a^{(n-1)}(M,p) = t^{(n-1)}(M,p)$.
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For these results, pseudo-convexity is not necessary.

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- **Conjecture:** When M is pseudo-convex, for $1 \leq s \leq n-1$, $a^{(s)}(M,p) = t^{(s)}(M,p) = c^{(s)}(M,p)$.

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- When $M \subset \mathbb{C}^3$, $a^{(1)}(M,p) = c^{(1)}(M,p)$.

Huang-Y. (2021): When M is pseudo-convex,

$$a^{(n-2)}(M,p) = t^{(n-2)}(M,p) = c^{(n-2)}(M,p).$$

In particular, this gives a complete solution for $n = 3$.

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(In this case, $a^{(1)}(M,p) = c^{(1)}(M,p)$ is due to Abdallah TALHAOUI (1983))

A Conjecture of D'Angelo (1986)

For a fixed tangent $(1,0)$ vector field L, as in $\mathbb{C}^2$, we can similarly define $t(L,p)$ and $c(L,p)$.

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D'Angelo Conjecture:

Let M be a pseudoconvex smooth hypersurface, $p \in M$. Then for any fixed $(1,0)$ tangent vector field L, we have $t(L,p) = c(L,p)$.

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It implies one equality of the Bloom Conjecture.

Progress on the D'Angelo Conjecture

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Recently, we have made some new progress on this problem.

Kohn's case (n=2)

WLOG, we assume $p = 0$. In $\mathbb{C}^2$ case, for any two $(1,0)$ tangent vectors L and L' with $L(0), L'(0) \neq 0$, we have $L = fL'$ with $f(0) \neq 0$. Hence

$$t(L,0) = t^{(1)}(M,0), \quad c(L,0) = c^{(1)}(M,0).$$

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The first approach is to achieve the equality via $a^{(1)}(M, 0)$. Namely, we prove

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Kohn's case (n=2)

In nornal form, We may assume

$$\rho = \mathsf{Im}(w) + P(z, \overline{z}) + O(|z|^{m+1} + |z\mathsf{Re}(w)| + |\mathsf{Re}(w)|^2).$$

Then $t(L, 0)$ and $c(L, 0)$ can be obtained by direct computation. In fact, if

$$L = \frac{\partial}{\partial z} - \rho_z(\rho_w)^{-1}\frac{\partial}{\partial w}.$$

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Then we have the following explicit formulas:

$$\lambda(L,\overline{L}) = \frac{1}{|\rho_w|^2}\Big\{\rho_{z\overline{z}}|\rho_w|^2 - 2\mathrm{Re}(\rho_{z\overline{w}}\rho_w\rho_{\overline{z}}) + r_{w\overline{w}}|\rho_z|^2\Big\}.$$

$$[\partial\rho, [\cdots[[L,\overline{L}], L_1], \cdots, L_{m-2}](0) = \frac{\partial^{r+s}}{\partial z^r \partial z^s}\rho(0).$$

Here $L_1, \cdots, L_{m-2} = L$ or $\overline{L}$, r and s are the numbers of L and $\overline{L}$.

Higher dimensional case

For Bloom-Graham's case, the proofs of $a^{(n-1)}(M,0) = t^{(n-1)}(M,0)$ and $a^{(n-1)}(M,0) = c^{(n-1)}(M,0)$ are more or less the same. We still have the normal form and explicit computation.

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When we deal with the Bloom Conjecture and the D'Angelo Conjecture in higher dimensional case, the pseudoconvex is necessary.

The difficulty of these problems lies in how to make use this pseudoconvex condition.

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For higher dimensional case ($n \geq 4$), the Bloom Conjecture itself is still unknown.

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For $L_1, \cdots, L_{k+1} = L$ or $\overline{L}$ and any tangent vector field L', define

$$\alpha_{L'} = \eta([T, L']),$$

and

$$\Gamma_{k+2} = [\cdots[[L, \overline{L}], L_1]\cdots, L_k].$$

If $L_k = L$, then

$$\eta(\Gamma_{k+2}) = \eta([\Gamma_{k+1}, L]) = (\alpha_L - L)\eta(\Gamma_{k+1}) - \lambda(L, \pi_{0,1}\Gamma_{k+1}).$$

If $L_k = \overline{L}$, then

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The crucial fact in $\mathbb{C}^2$ is that there always exists a function f such that

$$\pi_{1,0}\Gamma_{k+1} = fL, \ \pi_{0,1}\Gamma_{k+1} = \overline{gL}. \qquad (*)$$

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Thus by induction,

$$\eta(\Gamma_{k+2}) = \prod_{j=1}^{k}(\alpha_{L_j} - L_j)\lambda(L, \overline{L}) + P_{k-1}\lambda(L, \overline{L}).$$

P_j is a differential operator of order at most j along L and $\overline{L}$.

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   The second question is trivial if L is a real tangent vector field.

Direct connection for higher dimensional case

Write

$$\mathcal{L}^{m+2} = [\cdots[[L, \overline{L}], L_1]\cdots, L_m] \quad L_j = L \text{ or } \overline{L}.$$

Then if $L_m = L$

$$\eta(\mathcal{L}^{m+2}) = (\alpha_L - L)\eta(\mathcal{L}^{m+1}) - \lambda(L, \Pi_{0,1}\mathcal{L}^{m+1})$$

If if $L_m = \overline{L}$

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Hence by induction, we obtain

$$\eta(\mathcal{L}^{m+2}) = (-1)^m L_m \cdots L_1 \lambda(L, \overline{L}) + \mathcal{R}.$$

$\mathcal{R}$ is extremely complicated, it is no longer lower times derivative of $\lambda(L, \overline{L})$ along L and $\overline{L}$.

Write $X = \Pi_{1,0}[L, \overline{L}]$. In the case of $t = 4$ or $c = 4$,

$$\eta([[[L, \overline{L}], L], \overline{L}]) + \eta([[[L, \overline{L}], \overline{L}], L]) = (\overline{L}L + L\overline{L})\lambda(L, \overline{L}) + 2\lambda(X, \overline{X}).$$

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The key point is that both $(\overline{L}L + L\overline{L})\lambda(L, \overline{L})$ and the remainder term $\lambda(X, \overline{X})$ are positive, due to the pseudoconvexity.

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The key point is that both $(\overline{L}L + L\overline{L})\lambda(L, \overline{L})$ and the remainder term $\lambda(X, \overline{X})$ are positive, due to the pseudoconvexity.

It is not easy to achieve such a positive remainder term even for the degree 6 case.

Relation between these invariants

Example: Let $M \subset \mathbb{C}^4$ be a real hypersurface defined by

$$r = -2\mathrm{Im}w + |z_1|^4 + |z_1|^2|z_2|^2 + |z_1|^2|z_3|^2 + |z_2^2 - z_3^3|^4.$$

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The Caltin multitypes at 0 are $4,\ 4,\ 4$,

The Bloom regular contact types are $4,\ 8,\ 12$,

The D'Angelo finite types are $4,\ 8,\ +\infty$.

Thank you for your attention!