K-stability and Nevanlinna-Diophantine theory

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Canonical metrics and stability

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Calabi Conjecture. Given a compact Kähler manifold (X, ω) together with a 2-form R representing $c_1(X)$, one can always find a Kähler form $\hat{\omega}$ such that $[\omega] = [\hat{\omega}]$ and Ric $(\hat{\omega}) = R$.

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Valuative criterion of K-stability.

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The notion of the *K-stability* of Fano varieties is an algebro-geometric stability condition. An important problem in algebraic geometry is to find a simple criterion to test the K-stability of the variety X. One fundamental development is the equivalent description of the notions of K-stability, using the valuation over the function filed $K(X)$ ($\mathrm{ord}_E f$, where E is a irreducible divisor on X)

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Valuative criterion of K-stability. 1. X is uniformly K-stable (resp. semi-satble) if and only if $\delta(-K_X) > 1$ (resp. ≥ 1) (Fuji-Li).

2. X is K-stable if and only if $A_X(E) > \beta(-K_X, E)$ for any E (Blum-Xu).

K-stability through the base type divisor

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Blum-Jonsson (Blum-Jonsson, Advances in Math., 2020) used *m*-basis type to describe the stability threshold $\delta(L) = \inf_E \frac{A_X(E)}{\beta(L,E)}$. For m sufficient large, we say D is a *m*-basis type divisor if $D = \frac{1}{mN_m}((s_1) + \cdots + (s_{N_m}))$ where $\{s_1, \ldots, s_{N_m}\}$ forms a basis of $H^0(X, mL)$. Recall the algebraic geometry definition of "log canonical threshold": $\mathrm{lct}(D) = \min_E \frac{A_X(E)}{\mathrm{ord}_E(D)}$. Let $\delta_m(L) := \inf_D \mathrm{lct}(D) = \inf_D \min_E \frac{A_X(E)}{\mathrm{ord}_E(D)} = \inf_E \frac{A_X(E)}{S_m(E)}$, where $D \sim_{\mathbb{Q}} L$ runs through over all *m*-basis type divisors. Since $\lim_{m\to\infty} S_m(E) = \beta(L, E)$ (as we described earlier), Blum-Jonsson proved that $\lim_{m\to} \delta_m(L) = \delta(L)$. We note that this gives us a way to verify K-stability for explicit Fano varieties, by estimating $\delta_m(-K_X)$.

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This allows purely algebro-geometric proofs of Käher-Einstein metrics.

Nevanlinna theory

The Second Main Theorem(Nevanlinna, 1929).

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In 2004, Ru extended the above result to hypersurfaces for $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with Zariski dense image.
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Theorem (Ru, 2009). Let $f : \mathbb{C} \to X$ be holo and Zariski dense, $D_1, \dots, D_q$ be divisors in general position in X. Assume that $D_j \sim d_j A$ (A being ample). Then, for $\forall\ \epsilon > 0$,

$$\sum_{j=1}^q \frac{1}{d_j} m_f(r, D_j) \leq_{exc} (\dim X + 1 + \epsilon)T_{f,A}(r).$$

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In the case when $D_j \sim A$, then $\beta(D, D_j) = \frac{q}{n+1}$, where $D = D_1 + \cdots + D_q$.

Theorem (Ru-Vojta, 2020) [Arithmetic Part] Let X be a projective variety over a number field k, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle on X with $h^0(L^N) \geq 1$ for N big enough. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality

$$\sum_{i=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq (1+\epsilon) h_L(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

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where the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent. Note: The $D \sim_{\mathbb{Q}} L$ is of m-basis type if $D := \frac{1}{mN_m} \sum_{s \in \mathcal{B}} (s)$, where $\mathcal{B}$ is a basis of $H^0(X, \mathcal{L}^{\otimes m})$, where $N_m = \dim H^0(X, \mathcal{L}^{\otimes m})$.

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The proof is using the Basic Theorem (m-base estimate) by choosing a a suitable m-basis of $H^0(X, mL)$ through the filtration filtration $\mathcal{F}_m^t = H^0(X, mL - tE), t \geq 0$ of $H^0(X, mL)$

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