Proper mappings between indefinite hyperbolic spaces and type I classical domains

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Given integers $n \geq 2$ and $0 \leq l \leq n-1$, the generalized complex unit ball is defined as the following domain in $\mathbb{P}^n$:

$$\mathbb{B}_l^n = \{[z_0, ..., z_n] \in \mathbb{P}^n : |z_0|^2 + ... + |z_l|^2 > |z_{l+1}|^2 + ... + |z_n|^2\}.$$

For $0 \leq k \leq m$, let $I_{k,m}$ be the $m \times m$ diagonal matrix, where its first k diagonal elements equal -1 and the rest equal 1. Denote by $SU(l+1, n+1)$ the special indefinite unitary group that consists of matrices $A \in SL(n+1, \mathbb{C})$ satisfying $AI_{l+1,n+1}\overline{A}^t = I_{l+1,n+1}$.

The generalized ball $\mathbb{B}_l^n$ possesses a canonical indefinite metric $\omega_{\mathbb{B}_l^n}$ that is invariant under the action of its automorphism group $SU(l+1, n+1)$:

$$\omega_{\mathbb{B}_l^n} = -\sqrt{-1}\partial\bar{\partial}\log\left(\sum_{j=0}^{l}|z_j|^2 - \sum_{j=l+1}^{n}|z_j|^2\right).$$

The generalized ball equipped with the above indefinite metric is often called an indefinite hyperbolic space form.

Theorem (A,Baouendi-Ebenfelt-Huang, 2011)

Let $N \geq n$, $1 \leq l \leq \frac{n-1}{2}$, $1 \leq l' \leq \frac{N-1}{2}$ and $1 \leq l \leq l' < 2l$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}^n_l$ with $U \cap \mathbb{B}^n_l$ being connected, and F a holomorphic map from U into $\mathbb{P}^N$. Assume $F(U \cap \mathbb{B}^n_l) \subseteq \mathbb{B}^N_{l'}$ and $F(U \cap \partial\mathbb{B}^n_l) \subseteq \partial\mathbb{B}^N_{l'}$. Then F is an isometric embedding from $(U \cap \mathbb{B}^n_l, \omega_{\mathbb{B}^n_l})$ into $(\mathbb{B}^N_{l'}, \omega_{\mathbb{B}^N_{l'}})$.

Here we say F is isometric if it preserves the indefinite hyperbolic metrics: $F^*(\omega_{\mathbb{B}^N_{l'}}) = \omega_{\mathbb{B}^n_l}$ on $U \cap \mathbb{B}^n_l$.

M. S. Baouendi, P. Ebenfelt, X. Huang, *Holomorphic mappings between hyperquadrics with small signature difference*, Amer. J. Math. 133 (6) (2011) 1633-1661.

By using a different approach that utilizes structure of the moduli space of linear subspaces contained in generalized balls, Ng establishes the global version of Theorem.

Theorem (B,Ng, 2013)

Let $1 \leq l < \frac{n}{2}, 1 \leq l' < \frac{N}{2}$ *and* $f : \mathbb{B}^n_l \to \mathbb{B}^N_{l'}$ *be a proper holomorphic map. If* $l' \leq 2l - 1$, *then* f *extends to a linear embedding of* $\mathbb{P}^n$ *into* $\mathbb{P}^N$.

S. Ng, *Proper holomorphic mappings on flag domains of* $SU(p,q)-$*type on projective spaces*, Michigan Math. J., 62 (2013) 769-777.

Main result

Theorem (1)

Let $N \geq n \geq 3$, $1 \leq l \leq n-2, l \leq l' \leq N-1$. Let U be an open subset in $\mathbb{P}^n$ containing some $p \in \partial\mathbb{B}_l^n$ and F be a holomorphic map from U into $\mathbb{P}^N$. Assume $U \cap \mathbb{B}_l^n$ is connected and $F(U \cap \mathbb{B}_l^n) \subseteq \mathbb{B}_{l'}^N$, $F(U \cap \partial\mathbb{B}_l^n) \subseteq \partial\mathbb{B}_{l'}^N$. Assume one of the following conditions holds:

$$(1).\ \ l' < 2l, l' < n-1;$$

$$(2).\ \ l' < 2l, N-l' < n;$$

$$(3).\ \ N-l' < 2n-2l-1, l' < n-1;$$

$$(4).\ \ N-l' < 2n-2l-1, N-l' < n.$$

Then F is an isometric embedding from $(U \cap \mathbb{B}_l^n, \omega_{\mathbb{B}_l^n})$ to $(\mathbb{B}_{l'}^N, \omega_{\mathbb{B}_{l'}^N})$.

Definition (1)

Let F be a holomorphic rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$. Write $I \subseteq \mathbb{P}^n$ for the set of indeterminacy of F. We say F is a rational proper map from $\mathbb{B}^n_l$ to $\mathbb{B}^N_{l'}$, if F maps from $\mathbb{B}^n_l \setminus I$ to $\mathbb{B}^N_{l'}$ and maps $\partial\mathbb{B}^n_l \setminus I$ to $\partial\mathbb{B}^N_{l'}$.

Theorem (1) can be immediately applied to study rational proper maps between generalized balls.

Corollary

Let $N \geq n \geq 3$, $1 \leq l \leq n-2, l \leq l' \leq N-1$. *Assume one of the conditions in (1)–(4) of Theorem (1) holds. Let* F *be a rational proper map from* $\mathbb{B}^n_l$ *to* $\mathbb{B}^N_{l'}$. *Then* F *is a linear embedding from* $\mathbb{P}^n$ *to* $\mathbb{P}^N$. *Moreover, there exists* $h \in Aut(\mathbb{B}^N_{l'})$ *such that*

$$h \circ F([z]) = [z_0, ..., z_l, 0, ..., 0, z_{l+1}, ..., z_n, 0, ..., 0],$$

for $[z] = [z_0, ..., z_l, z_{l+1}, ..., z_n] \in \mathbb{P}^n$, *where the first zero tuple has* $l' - l$ *components.*

Note if $l \geq 1$, then every proper holomorphic map from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$ extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^N$ (see [Ng1]). Thus Corollary still holds if we assume F is a proper holomorphic from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$ instead of assuming it is a rational proper map from $\mathbb{B}_l^n$ to $\mathbb{B}_{l'}^N$.

Hence Corollary has Theorem (B) as its special case. It also has Corollary 1.6 in [BEH] as its special case.

Theorem (1) is optimal in the sense that it fails if none of the conditions (1)–(4) holds. Indeed, suppose all of the conditions (1)–(4) fail. Then one of the following two cases must hold:

- (A). $l' \geq 2l$ and $N - l' \geq 2n - 2l - 1$;
- (B). $N - l' \geq n$ and $l' \geq n - 1$.

The next two examples show the conclusion in Theorem (1) fails in each of the cases. Example (1) corresponds to the case (A) with $l' = 2l$ and $N - l' = 2n - 2l - 1$. Example (2) corresponds to the case (B) with $N - l' = n$ and $l' = n - 1$. Furthermore, the map in Example (1) is indeed a rational proper map between the generalized balls in the sense of Definition. Thus it also shows Corollary fails if none of the conditions (1)–(4) holds.

Example (1)

(Generalized Whitney map from $\mathbb{B}_l^{l+k}$ to $\mathbb{B}_{2l}^{2l+2k-1}$) Let $l \geq 1, k \geq 1$. Write $[w, z] = [w_0, w_1, \cdots, w_l, z_1, \cdots, z_k]$ for the homogeneous coordinates of $\mathbb{P}^{l+k}$ and

$$\mathbb{B}_l^{l+k} = \{[w, z] \in \mathbb{P}^{k+l} : \sum_{i=0}^{l} |w_i|^2 > \sum_{j=1}^{k} |z_j|^2\}.$$

Write $U = \mathbb{P}^{k+l} \setminus \{w_0 = z_k = 0\}$. Consider the following map $G : U \to \mathbb{P}^{2k+2l-1}$:

$$G([w, z]) = [w_0^2, w_0 w_1, \cdots, w_0 w_l, w_1 z_k, w_2 z_k, \cdots, w_l z_k, \\ w_0 z_1, w_0 z_2, \cdots, w_0 z_{k-1}, z_1 z_k, z_2 z_k, \cdots, z_{k-1} z_k, z_k^2].$$

Write the above components on the right hand side as $G_1, \cdots, G_{2k+2l}$ and set $|G|_{2l+1}^2 = -\sum_{i=1}^{2l+1} |G_i|^2 + \sum_{j=2l+2}^{2k+2l} |G_j|^2$. Notice that $|G|_{2l+1}^2 = (|w_0|^2 + |z_k|^2)(-\sum_{i=0}^{l} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Consequently, G maps $U \cap \mathbb{B}_l^{l+k}$ to $\mathbb{B}_{2l}^{2l+2k-1}$ and maps $U \cap \partial\mathbb{B}_l^{l+k}$ to $\partial\mathbb{B}_{2l}^{2l+2k-1}$. Hence the statement in Theorem (1) fails in this case.

Example (2)

(Generalized Whitney map from $\mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$) Let $l \geq 1, k \geq 1$. Let the homogeneous coordinates $[w, z]$ and $\mathbb{B}_l^{l+k} \subseteq \mathbb{P}^{l+k}$ be the same as in Example 6. Let $V = \mathbb{P}^{l+k} \setminus \{w_0 = w_l = 0\}$ and $H : V \to \mathbb{P}^{2k+2l-1}$ be defined as follows:

$$H([w, z]) = [w_0^2, w_0 w_1, \cdots w_0 w_{l-1}, w_l z_1, w_l z_2, \cdots, w_l z_k, \\ w_0 z_1, w_0 z_2, \cdots, w_0 z_k, w_1 w_l, w_2 w_l, \cdots, w_l^2].$$

Write the above components on the right hand side as $H_1, \cdots, H_{2k+2l}$ and set $|H|_{l+k}^2 = -\sum_{i=1}^{l+k} |H_i|^2 + \sum_{j=l+k+1}^{2k+2l} |H_j|^2$. Notice that $|H|_{l+k}^2 = (|w_0|^2 - |w_l|^2)(-\sum_{i=0}^{l} |w_i|^2 + \sum_{j=1}^{k} |z_j|^2)$. Thus H maps $V \cap \partial\mathbb{B}_l^{l+k}$ to $\partial\mathbb{B}_{l+k-1}^{2l+2k-1}$. In particular, set $V_+ := \{[w, z] \in V : |w_0| > |w_l|\}$. Then H maps $V_+ \cap \mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$ and maps $V_+ \cap \partial\mathbb{B}_l^{l+k}$ to $\partial\mathbb{B}_{l+k-1}^{2l+2k-1}$. Hence the statement in Theorem fails in this case. This map H is, however, not a rational proper map from $\mathbb{B}_l^{l+k}$ to $\mathbb{B}_{l+k-1}^{2l+2k-1}$ in the sense of Definition, as it maps some point in $\mathbb{B}_l^{l+k}$ to $\mathbb{P}^{2l+2k-1} \setminus \mathbb{B}_{l+k-1}^{2l+2k-1}$.

Lemma

Let l, m, a, b be nonnegative integers such that $m \geq 2, 1 \leq l \leq m-1$. Let $\varphi_1, ..., \varphi_a, \psi_1, ..., \psi_b$ be homogeneous holomorphic polynomials of the same degree in $\mathbb{C}^m$ such that

$$-\sum_{j=1}^{a} |\varphi_j(z)|^2 + \sum_{j=1}^{b} |\psi_j(z)|^2 = A(z, \bar{z})|z|_l^2, \ \ z \in \mathbb{C}^m, \tag{1}$$

where $A(z, \bar{z})$ is a real polynomial. Assume one of the following conditions holds:

$$(1). \ \ a < l, a < m-l;$$

$$(2). \ \ a < l, b < l;$$

$$(3). \ \ b < m-l, a < m-l;$$

$$(4). \ \ b < m-l, b < l.$$

Then $A(z, \bar{z}) \equiv 0$.

Recall for $0 \le l \le n-1$, the generalized Siegel upper-half space is defined by

$$\mathbb{S}_l^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) > \sum_{j=1}^{n-1} \delta_{j,l} |z_j|^2\}.$$

Its boundary is the standard hyperquadrics:
$\mathbb{H}_l^n = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathrm{Im}(w) = \sum_{j=1}^{n-1} \delta_{j,l} |z_j|^2\}$. Similarly for $l \le l' \le N-1$, we define

$$\mathbb{S}_{l,l',n}^N = \{(Z,W) \in \mathbb{C}^{N-1} \times \mathbb{C} : \mathrm{Im}(W) > \sum_{j=1}^{N-1} \delta_{j,l,l',n} |Z_j|^2\}.$$

And $\mathbb{S}_{l'}^N, \mathbb{H}_{l'}^N, \mathbb{H}_{l,l',n}^N$ are all defined in a similar manner. Now for $(z,w) = (z_1, \cdots, z_{n-1}, w) \in \mathbb{C}^n$, let $\Psi(z,w) = [i+w, 2z, i-w] \in \mathbb{P}^n$. Then Ψ is the Cayley transformation which biholomorphically maps $\mathbb{S}_l^n$ and its boundary $\mathbb{H}_l^n$ onto $\mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$ and $\partial \mathbb{B}_l^n \setminus \{[z_0, \cdots, z_n] : z_0 + z_n = 0\}$, respectively.

Composing F with automorphisms of $\mathbb{B}_l^n$ and $\mathbb{B}_{l'}^N$ if necessary, we assume that $q_0 = [1, 0, ..., 0, 1] \in \partial\mathbb{B}_l^n$ and $F(q_0) = [1, 0, ..., 0, 1] \in \partial\mathbb{B}_{l'}^N$. Denote by Ψ the aforementioned Cayley transformation from $\mathbb{S}_l^n$ to $\mathbb{B}_l^n$, and Φ the Cayley transformation from $\mathbb{S}_{l,l',n}^N$ to $\mathbb{B}_{l'}^N$. Then $\widetilde{F} := \Phi^{-1} \circ F \circ \Psi$ is well-defined in a small neighborhood of $0 \in \mathbb{H}_l^n$; and $\tilde{F}$ is side-preserving (i.e., it maps $\mathbb{S}_l^n$ to $\mathbb{S}_{l,l',n}^N$ near 0). Moreover, by the definition of the geometric rank (see Section 3 in [HLTX]), to show F is of geometric rank zero near q_0, it suffices to prove the new map $\widetilde{F}$ has zero geometric rank near 0.

X. Huang, J. Lu, X. Tang, M. Xiao, *Boundary characterization of holomorphic isometric embeddings between indefinite hyperbolic spaces*, Adv. Math., 374 (2020) 107388.

To keep notations simple, we will still write the new map as F instead of $\widetilde{F}$. That is, F is now a holomorphic map from a neighborhood V of $0 \in \mathbb{H}_l^n$ to $\mathbb{C}^N$, satisfying

$$F(V \cap \mathbb{S}_l^n) \subseteq \mathbb{S}_{l,l',n}^N \quad \text{and} \quad F(V \cap \mathbb{H}_l^n) \subseteq \mathbb{H}_{l,l',n}^N.$$

By shrinking V if necessary, we can additionally assume $M_1 := V \cap \mathbb{H}_l^n$ is connected and F is CR transversal along M_1.

Next for each $p \in M_1$, we associate it with a map F_p defined as:

$$F_p = \tau_p^F \circ F \circ \sigma_p^0. \tag{2}$$

Here $\sigma_p^0 \in \mathrm{Aut}(\mathbb{H}_l^n)$ and $\tau_p^F \in \mathrm{Aut}(\mathbb{H}_{l,l',n}^N)$ are as defined in [HLTX]. Then F_p is a holomorphic map in a neighborhood of $0 \in \mathbb{C}^n$, which sends an open piece of $\mathbb{H}_l^n$ into $\mathbb{H}_{l,l',n}^N$ with $F_p(0) = 0$. Moreover, $F_p(U \cap \mathbb{S}_l^n) \subseteq \mathbb{S}_{l,l',n}^N$. Let F_p^*, F_p^{**} be the first and second normalizations of F_p. Then, F_p^{**} map 0 to 0, and maps $\mathbb{H}_l^n$ (respectively, $\mathbb{S}_l^n$) to $\mathbb{H}_{l,l',n}^N$ (respectively, $\mathbb{S}_{l,l',n}^N$) near 0. Write $F_p^{**} = (f_p^{**}, \phi_p^{**}, g_p^{**})$, where f_p^{**} has $n-1$ components, ϕ_p^{**} has $N-n$ components, and g_p^{**} is a scalar function.

Lemma. *(Lemma 2.2 in [BH]) For each $p \in M_1$, F_p^{**} satisfies the normalization condition:*

$$\begin{cases} f_p^{**} = z + \frac{i}{2} a_p^{**(1)}(z) w + O_{wt}(4) \\ \phi_p^{**} = \phi_p^{**(2)}(z) + O_{wt}(3) \\ g_p^{**} = w + O_{wt}(5), \end{cases}$$

with

$$\langle \bar{z}, a_p^{**(1)}(z) \rangle_l |z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2, \ \tau = l' - l.$$

M. S. Baouendi, X. Huang, *Super-rigidity for holomorphic mappings between hyperqadrics with positive signature*, J. Differential Geom. 69 (2) (2005) 379-398.

We briefly recall the notion of geometric rank. If we write $a_p^{**(1)}(z) = z\mathcal{A}(p)$ for any $(n-1) \times (n-1)$ matrix $\mathcal{A}(p)$, then the geometric rank of F at p is defined as the rank of the matrix $\mathcal{A}(p)$. In particular, F have geometric rank zero at p if and only if $\mathcal{A}(p)$ is the zero matrix.

The rank of the $(n-1)\times(n-1)$ matric $\mathcal{A}(p) = -2i(\frac{\partial^2 (f_p^{**})_k}{\partial z_j \partial w}|_0)_{1<j,k<(n-1)}$ denoted by $Rk_{F(P)}$, is called the geometric rank fo F at p.

Proposition (1)

The map F has zero geometric rank near q_0 along $U \cap \partial \mathbb{B}_l^n$.

X. Huang, *On a linearity problem of proper holomorphic maps between balls in complex spaces of different dimensions*, J. Differential Geom., 51 (1999) 13-33.

Set $\mathcal{A}_p(z,\bar{z}) = \langle \bar{z}, a_p^{**(1)}(z)\rangle_l$, which by (19) is a real polynomial. Then it follows from (19) that

$$\mathcal{A}_p(z,\bar{z})|z|_l^2 = |\phi_p^{**(2)}(z)|_\tau^2.$$

Write $m = n-1$, $a = \tau = l' - l$, and $b = N - n - (l' - l)$. Note if one of the conditions (1)–(4) in Theorem (1) holds, then one of the conditions (1)–(4) holds in Lemma. Then by Lemma, we see $\mathcal{A}_p(z,\bar{z}) \equiv 0$, and thus F has geometric rank zero at p.

In the second part of the talk, we apply Theorem (1) and Corollary to study a mapping problem between type I classical domains. The study of proper holomorphic maps between bounded symmetric domains of high rank goes back to the work of Tumanov-Henkin [TH] (see also Henkin-Novikov [HN]). They proved that any proper self-mapping of an irreducible bounded symmetric domain of rank at least two is an automorphism. Since then, rigidity and classification problems for holomorphic proper maps between bounded symmetric domains have attracted much attention.

A. E. Tumanov, G. M. Henkin, Local characterization of analytic automorphisms of classical domains, (Russian) Dokl. Akad. Nauk SSSR 267(4) (1982) 796-799.

G. M. Henkin and R. Novikov, *Proper mappings of classical domains*, in Linear and Complex Analysis Problem Book, Lecture Notes in Math. Vol. 1043, Springer, Berlin (1984) 625-627.

Let $F : \Omega_1 \to \Omega_2$ be a proper holomorphic map between two bounded symmetric domains Ω_1 and Ω_2. Tsai [Ts] proved the total geodesy of F under the assumption that $\mathrm{rank}(\Omega_1) \geq \mathrm{rank}(\Omega_2) \geq 2$ and Ω_1 is irreducible. Much less is known about the remaining case when $\mathrm{rank}(\Omega_1) < \mathrm{rank}(\Omega_2)$ and the studies so far are mainly focused on the type I classical domains. Many interesting results along these lines can be found in [M1, T1, T2, Ng2]. We pause to recall the definition of the type I classical domains.

- I. Tsai, *Rigidity of proper holomorphic maps between symmetric domains*, J. Differential Geom., 37 (1993) 123-160.
- N. Mok, *Nonexistence of proper holomorphic maps between certain classical bounded symmetric domains*, Chinese Annals of Mathematics, Series B, 29 (2008) 135-146.
- Z. Tu, *Rigidity of proper holomorphic mappings between equidimensional bounded symmetric domains*, Proc. Amer. Math. Soc., 130 (2002) 1035-1042.
- Z. Tu, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains*, Mathematische Zeitschrift, (2002) 13-35.

Let r and s be positive integers. Write $M(r, s; \mathbb{C})$ for the set of all $r \times s$ complex matrices and I_s for the $s \times s$ identity matrix. The type I classical domain $D^I_{r,s}$ is defined by

$$D^I_{r,s} = \{Z \in M(r, s; \mathbb{C}) : I_s - \overline{Z}^t Z > 0\}.$$

An important step toward understanding proper maps between type I classical domains was due to Ng [Ng2] where he first found its deep connection with mapping problems for proper maps between generalized balls. In such a way, he would be able to apply results in CR geometry to mapping problems between bounded symmetric domains. Among other things, he proved that every proper holomorphic map $f : D^I_{r,s} \to D^I_{r',s}$ is standard (i.e., totally geodesically isometric embedding, up to normalization constants, with respect to the Bergman metrics) if $s \geq r \geq 2$ and $r' \leq \min\{2r - 1, s\}$.

In a recent nice paper of Chan [Ch], he posed the following conjecture, that was inspired by the work of Kim-Zaitsev [KZ2], Kim [K] and his own investigation in [Ch]. The statement in the conjecture would generalize the aforementioned theorem of Ng when $r' < s$.

S. T. Chan, *Rigidity of proper holomorphic maps between type-I irreducible bounded symmetric domains*, Int. Math. Res. Not., doi.org/10.1093/imrn/rnaa373.

S. Ng, *Holomorphic Double Fibration and the mapping problems of Classical Domains*, International Mathematics Research Notices, 2 (2015) 291-324.

S. Kim, D. Zaitsev, *Rigidity of CR maps between Shilov boundaries of bounded symmetric domains*, Invent. Math., 193 (2) (2013) 409-437.

S. Kim, *Proper holomorphic maps between bounded symmetric domains*, In complex analysis and geometry, (2015) 207-219.

S. Kim, D. Zaitsev, *Rigidity of proper holomorphic maps between bounded symmetric domains*, Math. Ann., 362 (1-2) (2015) 639-677.

Conjecture

Let $f : D^I_{p,q} \to D^I_{p',q'}, p \geq q > 1$ *be a proper holomorphic map. Assume* $q' < p$ *and one of the following conditions holds: (1)* $p' < 2p - 1$*; (2)* $q' < 2q - 1$. *Then*

(I). $p' \geq p,\ q' \geq q$.

(II). Moreover, after composing with suitable automorphisms of $D^I_{p,q}$ *and* $D^I_{p',q'}$, f *takes the following form:*

$$f : z \to \begin{pmatrix} z & 0 \\ 0 & h(z) \end{pmatrix}. \tag{3}$$

Here h *is a certain holomorphic* $(p' - p) \times (q' - q)-$*matrix valued function on* $D^I_{p,q}$ *satisfying that* $I_{q'-q} - \overline{h}^t h$ *is positive definite on* $D^I_{p,q}$.

In the remaining context of the paper, as in [Ch], if a proper map $f : D^I_{p,q} \to D^I_{p',q'}$ satisfies the conclusion of Conjecture (i.e., it takes the form (3) after composing with automorphisms), then we say f is of diagonal type. It is known that Conjecture holds under the additional assumption that f extends smoothly to a neighborhood of a smooth boundary point. This is a consequence of results obtained by Kim-Zaitsev [KZ2] and Kim [K]. (See Corollary 1 in [KZ2] for case (1), and Theorem 1.2 in [K] for case (2)). Moreover, the assumption in (1) or (2) of Conjecture cannot be weakened. Indeed, Seo (see page 445 in [S1]) constructed a proper holomorphic map (generalized Whitney map) from $D^I_{r,s}$ to $D^I_{2r-1,2s-1}$, which is not of diagonal type.

A. Seo, *New examples of proper holomorphic maps among symmetric domains*, Michigan Math. J. 64 (2) (2015) 435-448.

By further developing the double fibration ideas introduced in [Ng2], Chan himself [Ch] confirmed part (I) of Conjecture. He also proved part (II) of Conjecture under the condition in (2), while still left open part (II) under the condition in (1). See Theorem 1.3 in [Ch].

We give a complete affirmative answer to Conjecture under the condition in (1). Thus our result together with the work of Chan [Ch] leads to the following theorem:

Theorem (2)

Conjecture holds.

Let $f : D^l_{q,p} \to D^l_{q',p'}$ be a holomorphic map. Say f is fibral-image-preserving with respect to the double fibrations:

$$D_{q,p} \xleftarrow{\pi^1_{q,p}} \mathbb{P}^{q-1} \times D^l_{q,p} \xrightarrow{\pi^2_{q,p}} D^l_{q,p},$$

$$D_{q',p'} \xleftarrow{\pi^1_{q',p'}} \mathbb{P}^{q'-1} \times D^l_{q',p'} \xrightarrow{\pi^2_{q',p'}} D^l_{q',p'}.$$

Here

$$\pi^1_{q,p}([X], Z) = [X, XZ]_q, \quad \text{for } [X] \in \mathbb{P}^{q-1}, Z \in D^l_{q,p}.$$

And $\pi^2_{q,p}$ is the standard projection onto $D_{q,p}$.

For $x = [A, B]_q \in D_{q,p} \subseteq \mathbb{P}^{q+p-1}$ and $Z \in D^I_{q,p}$, their fibral images are defined, respectively, as the following:

$$x^\sharp = [A, B]^\sharp_q = \pi^2_{q,p}\big((\pi^1_{q,p})^{-1}([A, B]_q)\big) \subseteq D^I_{q,p}; \qquad Z^\sharp = \pi^1_{q,p}\big((\pi^2_{q,p})^{-1}(Z)\big) \subseteq D_{q,p}.$$

As shown in [Ng2], we indeed have the following formulas for the fibral images:

$$x^\sharp = \{Z \in D^I_{q,p} : AZ = B\}; \qquad Z^\sharp = \{[A, AZ]_q \in D_{q,p} : [A] \in \mathbb{P}^{q-1}\}.$$

If for any $[A, B]_q \in D_{q,p}$, we have $f([A, B]^\sharp_q) \subseteq [C, D]^\sharp_{q'}$ for some $[C, D]_{q'} \in D_{q',p'}$. Furthermore, let U be an open subset of $D_{q,p}$. Say a holomorphic map $g : U \to D_{q',p'}$ is a moduli map of f on U if $f([A, B]^\sharp_q) \subseteq g([A, B]_q)^\sharp$ for all $[A, B]_q \in U$.

Proposition (1)

Let $f : D^I_{q,p} \to D^I_{q',p'}$ *be a proper holomorphic map where* $p \geq q \geq 2$, *and* $3 \leq q' < p$. *Then the following statements hold.*

(a). Then f *is fibral-image-preserving with respect to the double fibrations. And there exists a holomorphic map* $g : U \subseteq D_{q,p} \to D_{q',p'}$ *such that* g *is a moduli map of* f *on* U, *where* U *is a dense open subset of* $D_{q,p}$. *Furthermore,* g *extends to a rational map from* $\mathbb{P}^{p+q-1}$ *to* $\mathbb{P}^{p'+q'-1}$. *Write* I *for the set of indeterminacy of* g, *we have*

$$g(\partial D_{q,p} \setminus I) \subseteq \partial D_{q',p'}.$$

And we have $p' \geq p, q' \geq q$.

(b). We have g *maps* $D_{q,p} \setminus I$ *to* $D_{q',p'}$. *Consequently,* g *is a rational proper map from* $D_{q,p}$ *to* $D_{q',p'}$.

(c). If either $p' < 2p - 1$ *or* $q' < 2q - 1$, *then* f *is of diagonal type.*

We prove the following consequence of [Ng2, Ch]. The result generalizes a theorem of Tu [T2]. Note when $r = s - 1$, Proposition 13 is reduced to Theorem 1.1 in [T2]. It also generalizes Theorem 1.3 of [Ng2] in the case $r' = s$.

Proposition (2)

Let $s > r \geq 2$. Then every proper holomorphic map $f : D^{I}_{r,s} \to D^{I}_{s,s}$ is standard. That is, f is a totally geodesic isometric embedding (up to normalization constants) with respect to the Bergman metrics.

- Z. Tu, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains*, Mathematische Zeitschrift, (2002) 13-35.
- S. Ng, *Holomorphic Double Fibration and the mapping problems of Classical Domains*, International Mathematics Research Notices, 2 (2015) 291-324.

Corollary

There exist no proper holomorphic mappings from $D^I_{r,s+1}$ to $D^I_{s,s}$ for $s \geq r \geq 2$.

Note Corollary fails if $r = 1$ and $s \geq 2$. Indeed there is a proper holomorphic map from $D^I_{1,s+1} = \mathbb{B}^{s+1}$ to $D^I_{s,s}$ (see [T2]).

The End

Thank you!