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Concavity property of minimal L^2 integrals

Zhitong Mi

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences

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Background

Let X be a complex manifold. Let φ be a plurisubharmonic function (psh for short) on X.

Recall that the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ is defined to be the sheaf of germs of holomorphic functions $(f, x) \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-\varphi}$ is locally integrable near x.

It is known that $\mathcal{I}(\varphi)$ is a coherent analytic sheaf (Nadel). Denote

$$\mathcal{I}_+(\varphi) := \cup_{\epsilon>0}\mathcal{I}((1+\epsilon)\varphi).$$

Demailly posed the so-called strong openness conjecture (SOC for short) on multiplier ideal sheaves, i.e.,

$$\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi).$$

Background

The SOC is equivalent to the following theorem which was proved by Guan-Zhou.

Theorem 1.1 (Guan-Zhou, Annals of math, 2015)

Let $\varphi < 0$ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. Suppose F is a holomorphic function on Δ^n, which satisfies

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. Then for some $r \in (0,1)$, there exists a number $p > 1$ such that

$$\int_{\Delta_r^n} |F|^2 e^{-p\varphi} d\lambda_n < +\infty.$$

Background

When $\mathcal{I}(\varphi) = \mathcal{O}$, the SOC degenerates to the openness conjecture (OC for short) which was posed by Demailly and Kollár.

- 2-dim case of the OC was proved by C. Favre and M. Jonsson by using algebraic method.
- 2-dim case of the SOC was proved by M. Jonsson and M. Mustaţă by using algebraic method.
- The OC was proved by B. Berndtsson by using complex Brunn-Minkowski inequalities.
- The SOC was proved by Guan-Zhou by using L^2 extension theorem movably and induction.
- Recently, Chenyang Xu completed the algebraic approach to the OC.

Berndtsson's effectiveness result of the OC

Recall that Berndtsson proved the openness conjecture by establishing an effectiveness result of the openness conjecture:

Theorem 1.2 (Berndtsson, arXiv:1305.5781)

Let φ be a negative plurisubharmonic function on the unit ball $\mathbb{B}^n$. Let $z_0 \in \mathbb{B}^n$ such that $|z_0| \le \frac{1}{2}$. Assume that $\int_{\mathbb{B}^n} e^{-\varphi} < +\infty$. If p satisfies

$$\frac{1}{400(p-1)} \ge \frac{\int_{\mathbb{B}^n} e^{-\varphi} d\lambda_n}{\inf\limits_{z \in \mathbb{B}^n_{\frac{1}{2}}} \frac{1}{K(z)}},$$

where $K(z)$ is the Bergman kernel on $\mathbb{B}^n$, then $e^{-p\varphi}$ is integrable near z_0.

Guan-Zhou's effectiveness result of the SOC

Continuing the solution of the SOC, Guan-Zhou established an effectiveness result of the SOC.

Let F be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$, and $\varphi < 0$ be a psh function on D. Let $z_0 \in D$ be a point.
Denote that

$$\|F\|_\varphi := \left(\int_D |F|^2 e^{-\varphi} d\lambda_n \right)^{\frac{1}{2}}$$

and

$$K_{\varphi,F}(z_0) := \frac{1}{\inf\{\|F_1\|_0^2 : (F_1 - F, z_0) \in \mathcal{I}_+(2c_{z_0}^F(\varphi)\varphi)_{z_0} \,\&\, F_1 \in \mathcal{O}(D)\}},$$

where $c_{z_0}^F(\varphi) := \sup\{c \geq 0 : |F|^2 e^{-2c\varphi}$ is L^1 on a neighborhood of $z_0\}$ is the jumping number.

Guan-Zhou's effectiveness result of the SOC

Theorem 1.3 (Guan-Zhou, Invent. Math., 2015)

Let C_1 *and* C_2 *be two positive constants. We consider the set of pairs* (F, φ) *satisfying:*
(1) $\|F\|_{\varphi}^2 \leq C_1$;
(2) $\frac{1}{K_{\varphi,F}(z_0)} \geq C_2$.
Then for any $p > 1$ *satisfying*

$$\left(\frac{1}{(p-1)(2p-1)}\right)^{\frac{1}{p}} > \frac{C_1}{C_2},$$

we have

$$(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}.$$

Minimal L^2 integrals on pseudoconvex domains in $\mathbb{C}^n$

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and $z_0 \in D$ be a point. Let $I \subset \mathcal{O}_{z_0}$ be an ideal. Let F be a holomorphic function on D.

Denote that

$$C_{F,I}(D) := \inf\left\{\int_D |F_1|^2 d\lambda_n : (F_1 - F, z_0) \in I \,\&\, F_1 \in \mathcal{O}(D)\right\}.$$

Note that $C_{F,\mathcal{I}_+(2c^F_{z_0}(\varphi)\varphi)_{z_0}}(D) = \frac{1}{K_{\varphi,F}(z_0)}$.

Guan established a sharp effectiveness result of the SOC by considering the minimal L^2 integrals $C_{F,I}(\{\varphi < -t\})$ on all sublevel sets of the weight φ.

Guan's sharp effectiveness result of the SOC

Theorem 1.4 (Guan, Adv. in Math., 2019)

Let C_1 and C_2 be two positive constants. We consider the set of pairs (F,φ) satisfying:
(1) $\|F\|^2_\varphi \le C_1$;
(2) $\frac{1}{K_{\varphi,F}(z_0)} \ge C_2$.
Then for any $p > 1$ satisfying

$$\theta(p) > \frac{C_1}{C_2},$$

we have $(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}$, where $\theta(p) = \frac{p}{p-1}$, which is sharp.

Denote

$$G(t) := C_{F,\mathcal{I}(\varphi)_{z_0}}(\{\varphi < -t\}).$$

Guan established a concavity property of minimal L^2 integrals on pseudoconvex domains in $\mathbb{C}^n$.

Theorem 1.5 (Guan, Adv. in Math., 2019)

If $G(0) < +\infty$, then $G(-\log r)$ is concave with respect to $r \in (0, 1]$.

General minimal L^2 integrals

- Let M be a weakly pseudoconvex Kähler manifold. Let K_M be the canonical line bundle on M.
- Let ψ be a plurisubharmonic function on M, and let φ be a Lebesgue measurable function on M, such that $\psi+\varphi$ is a plurisubharmonic function on M. Denote $T=-\sup\limits_M \psi$.
- Let Z_0 be a subset of $\{\psi=-\infty\}$ such that $Z_0 \cap Supp(\mathcal{O}/\mathcal{I}(\varphi+\psi)) \neq \emptyset$.
- Let $U \supset Z_0$ be an open subset of M, and let f be a holomorphic $(n,0)$ form on U.
- Let $\mathcal{F} \supset \mathcal{I}(\varphi+\psi)|_U$ be a coherent subsheaf of $\mathcal{O}$ on U.

General minimal L^2 integrals

Definition

Define the function $G(t; c, \varphi, \psi, f, \mathcal{F}) : [T, +\infty) \to [0, +\infty]$ *as follows,*

$$\inf \left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right.$$
$$\left. \& \, \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\},$$

where c *is a nonnegative measurable function on* $(T, +\infty)$,
$|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$ *for any* $(n, 0)$ *form* f *and*
$(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ *means* $(\tilde{f} - f, z_0) \in (\mathcal{O}(K_M) \otimes \mathcal{F})_{z_0}$ *for all* $z_0 \in Z_0$.

Examples of minimal L^2 integrals

We consider the following example.

Let $M = \Omega$ be an open Riemann surface with nontrivial Green function $G_\Omega(z, w)$. Let $Z_0 = z_0$ be a point in Ω. Let (U, z) be a local coordinate system of z_0. Let $f = dz$ on U. Set $\psi = 2G_\Omega(z, z_0)$ and $\varphi = 0$. Set $\mathcal{F} = \mathcal{I}(\psi)|_U$. Let $c(t) \equiv 1$. At this time, we have

$$G(t; c, \varphi, \psi, f, \mathcal{F}) = \frac{2}{K_{\Omega_t}(z_0)},$$

where $K_{\Omega_t}(z_0)$ is the Bergman kernel on $\Omega_t := \{2G_\Omega(z, z_0) < -t\}$.

We introduce the following notation.

Definition

Let $c(t)$ be a nonnegative measurable function on $(T, +\infty)$.

$$\mathcal{H}^2(t; c, \varphi, \psi, f, \mathcal{F}) := \{\tilde{f} : \ \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)), \\ (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}), \\ \& \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty.\}$$

where $t \geq T$ is a real number.

We simply denote $G(t; c, \varphi, \psi, f, \mathcal{F})$ by $G(t)$ and $\mathcal{H}^2(t; c, \varphi, \psi, f, \mathcal{F})$ by $\mathcal{H}^2(t)$ when there is no misunderstanding.
If $\mathcal{H}^2(t) = \emptyset$, we set $G(t) = +\infty$.

We introduce a class of twisted factors $c(t)$ (so-called "**gain**" functions) as follows.

Definition

We call a positive measurable function $c(t)$ on $(T, +\infty)$ in class $\mathcal{G}_T$ if the following three statements hold:
(1) $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$;
(2) $c(t)e^{-t}$ *is decreasing with respect to* t;
(3) *for any compact subset* $K \subset M$, $e^{-\varphi}c(-\psi)$ *has a positive lower bound on* K.

We obtain the following concavity of $G(t)$ on M.

Theorem 2.1 (Guan-Mi, Sci China Math, 2022)

Let $c \in \mathcal{G}_T$ and smooth on $(T, +\infty)$. If there exists $t \in [T, +\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, where $h(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1, t \in [T, +\infty)$.

- We generalize Theorem 2.1 to the case $c(t)$ is only assumed to be Lebesgue measurable function in [Guan-Mi-Yuan, Concavity *II*, Researchgate].

Example

Let $M = \Delta \subset \mathbb{C}$ be the unit disc and $Z_0 = 0$ be the origin. Let $\psi = 4\log|z|$, $\varphi = 0$ and $c(t) \equiv 1$. Let $f = dz$ on Δ_ϵ, for some $\epsilon > 0$ small. Let $\mathcal{F}_0 = \mathcal{I}(\psi)_0 = (z^2)_0$. Then we have

$$G(t) = \inf\{\int_{\{4\log|z|<-t\}} |\tilde{f}|^2 : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \\ \&(\tilde{f} - f)_0 \in (z^2)_0\}.$$

By using the polar coordinate change, we know that the infimum is obtained by $\tilde{f} = dz$. Direct calculation shows that

$$G(t) = 2\pi e^{-\frac{t}{2}}.$$

At this time, $r = h(t) = \int_t^{+\infty} e^{-t_1} dt_1 = e^{-t}$ and $t = h^{-1}(r) = -\log r$. Hence $G(h^{-1}(r)) = 2\pi r^{\frac{1}{2}}$ which is concave with respect to r.

We introduce some properties of minimal L^2 integral function $G(t)$.

The following lemma is a characterization of $G(t) = 0$, where $t \geq T$.

Lemma 2.2 (Guan-Mi, Sci China Math, 2022)

The following two statements are equivalent:
(1) $f \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$.
(2) $G(t) = 0$.

The following lemma shows that there exists a unique holomorphic $(n,0)$ form related to $G(t)$.

Lemma 2.3 (Guan-Mi, Sci China Math, 2022)

Assume that $G(t)<+\infty$ for some $t\in[T,+\infty)$. There exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)=G(t)$$

and $(F_t-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$.
Furthermore, for any holomorphic $(n,0)$ form $\hat{F}$ on $\{\psi<-t\}$ satisfying $\hat{F}\in\mathcal{H}^2(t)$, we have the following equality

$$\int_{\{\psi<-t\}}|F_t|^2e^{-\varphi}c(-\psi)+\int_{\{\psi<-t\}}|\hat{F}-F_t|^2e^{-\varphi}c(-\psi)$$
$$=\int_{\{\psi<-t\}}|\hat{F}|^2e^{-\varphi}c(-\psi).$$

Properties of minimal L^2 integrals $G(t)$

The following lemma shows the lower semicontinuity of $G(t)$.

Lemma 2.4 (Guan-Mi, Sci China Math, 2022)

$G(t)$ is decreasing with respect to $t \in [T, +\infty)$, such that $\lim_{t \to t_0 + 0} G(t) = G(t_0)$ for any $t_0 \in [T, +\infty)$, and if $G(t) < +\infty$ for some $t > T$, then $\lim_{t \to +\infty} G(t) = 0$. Especially, $G(t)$ is lower semicontinuous on $[T, +\infty)$.

By using Lemma 2.3, Lemma 2.4 and then calculating the derivatives of $G(t)$, we can prove the Main Theorem.

Let M be an n-dimensional weakly pseudoconvex Kähler manifold. Let dV_M be a continuous volume form on M with no zero points. Let Y be a closed complex submanifold of X with codimension l.

Let $\psi < 0$ be a plurisubharmonic function on X which satisfies

(1) $Y \subset \{\psi = -\infty\}$, where $\{\psi = -\infty\}$ is a closed subset of X.

(2) For any $x \in Y$, there exists a local coordinate $(V; z_1, \cdots, z_n)$ of x such that $Y \cap V = \{z_{n-l+1} = \cdots = z_n = 0\}$, and

$\psi(z) = l \log \sum\limits_{n-l+1}^{n} |z_j|^2 + u(z)$ on V, where $u \in C^\infty(V)$.

The set of such polar functions ψ will be denoted by $A(Y)$.

Minimal L^2 integrals and optimal L^2 extension theorem

Following Ohsawa (Nagoya Math. J., 2001.), given a function $\psi \in A(Y)$ on M, one can associate a positive measure $dV_M[\psi]$ on Y as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_Y f d\mu \geq \limsup_{t\to+\infty} \frac{l}{\sigma_{2l-1}} \int_M f e^{-\psi} \mathbb{I}_{\{-t-1<\psi<-t\}} dV_M$$

for any nonnegative continuous function f with $suppf \subset\subset M$, where $\mathbb{I}_{\{-t-1<\psi<-t\}}$ is the characteristic function of the set $\{-1-t<\psi<-t\}$. Here σ_m is the volume of the unit sphere in $\mathbb{R}^{m+1}$.

Let $\psi \in A(Y)$ be a plurisubharmonic function on M. Let $\Omega \subset\subset M$ be an open subset of M. We assume that $\psi < -T$ on Ω. Let φ be a smooth plurisubharmonic functions on $\overline{\Omega}$.

Let U be an open neighborhood of $Y \cap \Omega$ in Ω and $f \in H^0(U, \mathcal{O}(K_M))$, which satisfies

$$\int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_M[\psi] < +\infty.$$

Let $c(t) \in \mathcal{G}_T$ be smooth, then one can define the function $G_\Omega(t; c)$ on Ω by

$$\begin{aligned}\inf\{\int_{\{\psi<-t\}\cap\Omega} &|\tilde{f}|^2 e^{-\varphi} c(-\psi) dV_M : \tilde{f} \in H^0(\{\psi < -t\} \cap \Omega, \mathcal{O}(K_M))\\ &\&(\tilde{f} - f) \in H^0(Y \cap \Omega, \mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)|_{Y\cap\Omega})\}\end{aligned}$$

Proposition 3.1 (Guan-Mi, Sci China Math, 2022)

Let $M, Y, \Omega, \psi, \varphi, f$ be as above. Then for any smooth positive function $c(t)$ in $\mathcal{G}_T$, the following equality holds

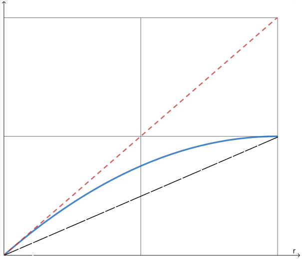
$$\lim_{t\to+\infty} \frac{G_\Omega(t;c)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} = \frac{\pi^l}{l!}\int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_M[\psi].$$

Remark 3.1 (Guan-Mi, Sci China Math, 2022)

Combining with the Theorem 2.1 and Proposition 3.1, we know

$$\frac{G_\Omega(T;c)}{\int_T^{+\infty} c(t_1)e^{-t_1}dt_1} \le \frac{\pi^l}{l!}\int_{Y\cap\Omega} |f|^2 e^{-\varphi} dV_M[\psi]. \quad (1)$$

It follows from Lemma 2.3 that there exists a holomorphic $(n,0)$ *form* F_Ω *on* Ω *such that* $G_\Omega(T;c) = \int_\Omega |F_\Omega|^2 e^{-\varphi} c(-\psi) dV_M$, *then* (1) *shows that* F_Ω *is a holomorphic extension of* f *from* Y *to* Ω *which satisfies the optimal* L^2 *estimate.*



Background: Suita conjecture

Let Ω be an open Riemann surface which admits a nontrivial Green function $G_\Omega(z,w)$.
Let (V_{z_0}, w) be a local coordinate neighborhood of z_0 satisfying $w(z_0) = o$ and $G_\Omega(z, z_0) = \log|w| + u(w)$ on V_{z_0}, where $u(w)$ is a harmonic function on V_{z_0}.
Let $c_\beta(z)$ be the logarithmic capacity which is locally defined by

$$c_\beta(z_0) := \exp(\lim_{z\to z_0} G_\Omega(z, z_0) - log|w(z)|).$$

Let κ_Ω be the Bergman kernel of holomorphic $(1,0)$ form on Ω. We define

$$B_\Omega(z)|dw|^2 := \kappa_\Omega|_{V_{z_0}}.$$

Background: Suita conjecture

Suita posed the following conjecture.

Suita conjecture (Arch. Rational Mech. Anal., 1972)

$\left(c_\beta(z_0)\right)^2 \le \pi B_\Omega(z_0)$ holds. Equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

- The inequality part of the Suita conjecture for bounded planar domains was solved by Błocki.
- The original form of the inequality part of the Suita conjecture was proved by Guan-Zhou.
- The equality part of the Suita conjecture was proved by Guan-Zhou, which completed the proof of the Suita conjecture.

Background: Suita conjecture

Let $M = \Omega$ be an open Riemann surface with nontrivial Green function $G_\Omega(z,w)$. Let $Z_0 = z_0$ be a point in Ω. Let (V,w) be a local coordinate system of z_0. Let $f = dw$ on V. Set $\psi = 2G_\Omega(z,z_0)$ and $\varphi = 0$. Set $\mathcal{F} = \mathcal{I}(\psi)|_U$. Let $c(t) \equiv 1$.

- By definition, we have

$$G(t; c, \varphi, \psi, f, \mathcal{F}) = \frac{2}{B_{\Omega_t}(z_0)},$$

  where $B_{\Omega_t}(z_0)$ is the Bergman kernel on $\Omega_t := \{2G_\Omega(z,z_0) < -t\}$.

- Denote $Y = z_0$. By direct calculation, we have

$$\int_Y |f|^2 dV_\Omega[\psi] = \frac{2\pi}{(c_\beta(z_0))^2}.$$

Hence $\left(c_\beta(z_0)\right)^2 = \pi B_\Omega(z_0)$ holds if and only if

$$G(0; c, \varphi, \psi, f, \mathcal{F}) = \pi \int_Y |f|^2 dV_\Omega[\psi]. \tag{2}$$

By Theorem 2.1 and Proposition 3.1, we know that equality (2) holds if and only if $G(-\log r; c, \varphi, \psi, f, \mathcal{F})$ is linear with respect to $r \in [0, 1]$.

Hence $\left(c_\beta(z_0)\right)^2 = \pi B_\Omega(z_0)$ holds if and only if $G(-\log r; c, \varphi, \psi, f, \mathcal{F})$ is linear with respect to $r \in [0, 1]$.

Linear case

As a corollary of Main Theorem, we give a necessary condition for the concavity degenerating to linearity.

Corollary 3.2 (Guan-Mi-Yuan, Concavity *II*, Researchgate)

Let $c(t) \in \mathcal{G}_T$, if $G(t;c) \in (0,+\infty)$ for some $t \geq T$ and $G(\hat{h}^{-1}(r);c)$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds)$, where $\hat{h}(t) = \int_t^{+\infty} c(l)e^{-l}dl$, then there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ and $G(t;c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq T$.
Furthermore

$$\int_{\{-t_1 \leq \psi < -t_2\}} |F|^2 e^{-\varphi} a(-\psi) = \frac{G(T;c)}{\int_T^{+\infty} c(t)e^{-t}dt} \int_{t_2}^{t_1} a(t)e^{-t}dt$$

for any nonnegative measurable function a on $(T,+\infty)$, where $T \leq t_2 < t_1 \leq +\infty$.

Linear case for various $c(t)$

It follows from Corollary 3.2 that we have the following remark for the linearity of $G(t;c)$ for various $c(t)$.

Remark 3.3 (Guan-Mi-Yuan, Concavity *II*, Researchgate)

If $\mathcal{H}^2(\tilde{c},t_0)\subset\mathcal{H}^2(c,t_0)$ for some $t_0\ge T$, we have

$$G(t_0;\tilde{c})=\int_{\{\psi<-t_0\}}|F|^2e^{-\varphi}\tilde{c}(-\psi)=\frac{G(T;c)}{\int_T^{+\infty}c(t)e^{-t}dt}\int_{t_0}^{+\infty}\tilde{c}(s)e^{-s}ds, \tag{3}$$

where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$.

Linear case for various φ

We now consider the relation between the linearity of $G(t;\varphi)$ and the function φ. We have the following necessary condition for the linearity of $G(t;\varphi)$ with respect to the weight function φ.

Corollary 3.4 (Guan-Mi, PKMJ, 2022)

If there exists a Lebesgue measurable function $\tilde{\varphi}$ such that $\psi+\tilde{\varphi}$ is a plurisubharmonic function on M and satisfies
(1) There exists constant $C_1, C_2 > T$ such that

$$\tilde{\varphi}|_{\{\psi<-C_1\}\cup\{\psi\geq -C_2\}} = \varphi|_{\{\psi<-C_1\}\cup\{\psi\geq -C_2\}}.$$

(2) $\tilde{\varphi}\geq\varphi$ on M and $\tilde{\varphi}>\varphi$ on an open subset U of M.
(3) $\tilde{\varphi}-\varphi$ is bounded on M.
Then $G(h^{-1}(r);\varphi)$ can not be linear with respect to $r\in(0,\int_T^{+\infty}c(t)e^{-t}dt]$.

It follows from Corollary 3.4 that we have

Corollary 3.5 (Guan-Mi, PKMJ, 2022)

If $\varphi + \psi$ is a plurisubharmonic function on M and $\varphi + \psi$ is strictly plurisubharmonic at some point $z_0 \in M$. Then $G(h^{-1}(r); \varphi)$ can not be linear with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$.

Characterizations for 1-dimensional case

Let Ω be an open Riemann surface which admits a nontrivial Green function $G_\Omega(z, w)$. let $Z_0 = z_0$ be a single point of Ω.
Let $\psi = kG_\Omega(z, z_0)$, where $k \geq 2$ is a real number.
Let φ be a subharmonic function on Ω. Let U be an open neighborhood of z_0 in Ω and f be a holomorphic $(1, 0)$ form on U. Let $\mathcal{F} = \mathcal{I}(\psi + 2\varphi)|_U$.
Let $c(t) \in C^\infty[0, +\infty)$ and $c(t) \in G_0$. Then we can define the function $G(t; c, 2\varphi)$.
We have the following necessary conditions for the minimal L^2 integrals $G(h^{-1}(r); c, 2\varphi)$ to be linear with respect to $r \in (0, \int_0^{+\infty} c(t_1)e^{-t_1}dt_1]$.

Theorem 3.6 (Guan-Mi, PKMJ, 2022)

Assume that $0 < G(0; c, 2\varphi) < +\infty$. If $G(h^{-1}(r); c, 2\varphi)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t_1)e^{-t_1}dt_1]$, then $\varphi = \log|g| + u$, where g is a holomorphic function on Ω and u is a harmonic function on Ω.

Characterizations for 1-dimensional case

Let $p : \Delta \to X$ be the universal covering from unit disc Δ to Ω.
We recall the following notations.

- A character χ on $\pi_1(\Omega)$ is a homomorphism from $\pi_1(\Omega)$ to $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ which takes values in the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
- We call the holomorphic function f on Δ is a multiplicative function if there is a character χ on $\pi_1(X)$, such that $g^* f = \chi(g) f$ for every $g \in \pi_1(\Omega)$ which naturally acts on the universal covering of Ω. Denote the set of such kinds of f by $\mathcal{O}^{\chi}(\Omega)$.
- For any harmonic function u on Ω, there exists a character χ_u associated to u and a multiplicative function $f_u \in \mathcal{O}^{\chi_u}(X)$, such that $|f_u| = p^* e^u$.
- For Green function $G_X(\cdot, z_0)$, one can find a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}| = p^* e^{G_\Omega(\cdot, z_0)}$.

We recall the following notations.

- Let $\kappa_{\Omega,\rho}$ be the weighted Bergman kernel with weight ρ of holomorphic $(1,0)$ form on a Riemann surface Ω.
- Let $B_{\Omega,\rho}(z)|dz|^2 := \kappa_{\Omega,\rho}$.
- Let u be a harmonic function on Ω and $\rho = e^{-2u}$.

Yamada posed the following extended Suita conjecture.

Extended Suita conjecture (Sūrikaisekikenkyūsho Kōkyūroku No. 1067, 1998)

$c_\beta^2(z_0) \leq \pi\rho(z_0)B_{\Omega,\rho}(z_0)$, and the equality holds if and only if $\chi_{-u} = \chi_{z_0}$.

The extended Suita conjecture was proved by Guan-Zhou.

Characterizations for single point case

Let $\psi = 2G_\Omega(z, z_0)$ in the definition of $G(t; c, 2\varphi)$. Let $c(t) \in C^\infty[0, +\infty)$ and $c(t) \in \mathcal{G}_0$. Using Theorem 3.6 and the solution of extended Suita conjecture by Guan-Zhou, we have the following characterization for $G(h^{-1}(r); c, 2\varphi)$ to be linear.

Theorem 3.7 (Guan-Mi, PKMJ, 2022)

Assume that $0 < G(0; c, 2\varphi) < +\infty$. *The minimal* L^2 *integral function* $G(h^{-1}(r); c, 2\varphi)$ *is linear with respect to* r *if and only if the following statements hold:*
(1) $\varphi = \log|g| + u$, *where* f_φ *is a holomorphic function on* X *and* u *is a harmonic function on* X.
(2) $\chi_{-u} = \chi_{z_0}$, *where* χ_{-u} *is the character associated to the function* $-u$.

References

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Thank you for listening!