Volume of a line bundle on a non-compact complex manifold

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Background

M: compact complex manifold, $\dim_{\mathbb{C}} M = n$.
$(L, h^L) \to M$: holomorphic line bundle. h^L: Hermitian fiber metric.
Locally, $h^L = e^{-\varphi}, \varphi \in C^\infty(U, \mathbb{R})$.
$\mathcal{R}^L$: Chern curvature of (L, h^L), $\mathcal{R}^L = i\partial\overline{\partial}\varphi$.
If $\mathcal{R}^L \geq 0$ (or > 0), L is called a semi-positive (or positive) line bundle.
For $k \in \mathbb{Z}, k > 0$, consider $L^k := L^{\otimes k}$ and
$\Omega^{0,q}(M, L^k) := C^\infty(M, \Lambda^q T^{*0,1} M \otimes L^k)$.
q-th Dolbeault cohomology group:

$$H^q(M, L^k) := \frac{\mathrm{Ker}\overline{\partial} : \Omega^{0,q}(M, L^k) \to \Omega^{0,q+1}(M, L^k)}{\mathrm{Im}\overline{\partial} : \Omega^{0,q-1}(M, L^k) \to \Omega^{0,q}(M, L^k)}.$$

For $q = 0$, $H^0(M, L^k)$: the space of global holomorphic sections. The volume of L is defined by

$$v(L) := \limsup_{k\to\infty} \frac{n!}{k^n} \dim H^0(M, L^k).$$

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For $q = 0$, $H^0(M, L^k)$: the space of global holomorphic sections. The volume of L is defined by

$$v(L) := \limsup_{k \to \infty} \frac{n!}{k^n} \dim H^0(M, L^k).$$

- Kodaira vanishing theorem(1950s): If L is positive, then $H^q(M, L^k) = 0$ for $q \geq 1$ when k is large.
- If L is positive, then

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty,$$

i.e., L is big.

- Kodaira embedding theorem(1950s): If L is positive, then $M \hookrightarrow \mathbb{C}P^n$ by the holomorphic sections of L^k when k is large.
- **Grauert-Riemenschneider conjecture** (1970): If L is semi-positive and positive at a point of M then L is big.
- L is big$\Rightarrow M$ is Moishezon manifold.

The Grauert-Riemschneider conjecture was first solved by Siu(1984) by using the $\overline{\partial}$-method. Then Demailly(1985) using his holomorphic Morse inequalities gave another proof of this conjecture.

$M(\Subset M')$: a relatively compact domain in complex manifold M' with smooth boundary.

$M := \{\rho < 0\}$, $\rho \in C^\infty(M')$ is a defining function for M.

$\mathcal{L} = i\partial\overline{\partial}\rho|_{T^{1,0}(\partial M)}$: Levi-form on M.
If $\mathcal{L} \geq 0$, M is called wealky pseudoconvex. If $\mathcal{L} < 0$, M is called strongly pseudoconcave.

$(L, h) \to M'$: holomorphic line bundle.

$H^0(M, L^k)$: The space of holomorphic sections of L^k.

Problem: When $\dim H^0(M, L^k) \gtrsim k^n, k \to \infty$?

Motivation

Extending Demailly's holomorphic Morse inequalities to non-compact complex manifolds:

- Bouche(1989), q-convex manifold, semi-positive holomorphic line bundle.
- Marinescu(1996), q-concave manifold, semi-negative holomorphic line bundle.
- Berman(2005), complex manifold with compact smooth boundary satisfying $Z(q)$-condition, no assumptions on the curvature of the line bundle.

After some integration assumption on the curvature of the line bundle, they got the 'big' property of line bundle on non-compact complex manifolds.

Conjecture(Henkin, Marinescu(96))
Let $M(\Subset M')$ be a strongly pseudoconcave complex manifold of $\dim M = n, n \geq 3$. Let $(L,h) \to M'$ be a positive line bundle. Then

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty.$$

Example:
Y: a compact complex manifold, $\dim_{\mathbb{C}} Y = n$.
$(L,h) \to Y$: positive holomorphic line bundle. Then $\dim H^0(Y, L^k) \gtrsim k^n, k \to \infty$.
Take $M := Y \setminus \overline{\mathbf{B}^n}$ where $\mathbf{B}^n$ is a Euclidean unit ball.
Then M is strongly pseudoconcave and

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty.$$

Motivation

Theorem(Marinescu, 1996)
Let $M(\Subset M')$ be a strongly pseudoconcave manifold of dimension $n, n \geq 3$. Let $(L, h) \to M'$ be a positive holomorphic line bundle. If

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty,$$

then there is a compact Moisezon manifold $\widehat{M}$ which contains M as an open set.

Motivation

Theorem (Hsiao-L., Asian J. Math., 2018; Herrmann-Hsiao-L., Math. Res. Lett., to appear)

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n-1, n \geq 2$. Let $(L,h) \to X$ be a positive CR line bundle. Assume that X admits a transversal CR S^1-action(or $\mathbb{R}$-action) which can be lifted to L. Then L is big, i.e.,

$$\dim H_b^0(X, L^k) \gtrsim k^n, k \to \infty.$$

Question: Can one extend CR sections of a holomorphic line bundle to holomorphic sections?

Kohn-Rossi(1965) If there is a non-trivial holomorphic function on M and the Levi-form of M has at least one positive eigenvalue everywhere, then all the CR sections of $L|_{\partial M}$ can be meromorphically extended into M.

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Reduction to the boundary

- $M \Subset M'$: a relatively compact domain with smooth boundary in a complex manifold M'.
- $X := \partial M$, $T^{1,0}X := T^{1,0}M' \cap \mathbb{C}TX$. $(X, T^{1,0}X)$ is a CR manifold.
- $(L, h) \to M'$: holomorphic line bundle.
- $L_X := L|_X$ is a CR line bundle over X.
- $H_b^0(X, L_X)$: the space of smooth CR sections of L_X.

Reduction to the boundary

- Assume M' admits a holomorphic $\mathbb{R}$-action.
- $T \in C^\infty(M', TM')$: the infinitesimal generator of the action.
- Suppose the $\mathbb{R}$-action preserves the boundary of M and satisfies the following transversal property

$$\mathbb{C}T_xX = \mathbb{C}T(x) \oplus T^{1,0}X \oplus T^{0,1}X, \forall x \in X.$$

Observation

- Suppose $(L, h^L) \to M'$ is positive.
- Assume the $\mathbb{R}$-action can be lifted to L.
- Let s be a local $\mathbb{R}$-invariant holomorphic trivializing section of L, $|s|^2 = e^{-2\phi}$.
- Since h^L is $\mathbb{R}$-invariant, then $JT(\phi)$ is globally well defined, where J is the complex structure on M'.

- $\mathcal{R}^{L_X}$: Chern curvature of the CR line bundle $L_X := L|_X$.
- $\mathcal{R}^{L_X}_x = \mathcal{R}^L|_{T^{1,0}X} + 2JT(\phi)(x)\mathcal{L}_x, \forall x \in X$.
- $\mathcal{R}^{L_X}$ could be degenerate even $\mathcal{R}^L$ is positive.

- In joint works with Herrmann, Hsiao and Marinescu, we see that if

$$\mathcal{R}^{L_X} + c\mathcal{L} > 0 \text{ on } T^{1,0}X$$

for some $c > 0$, we established the 'big' property of L_X.

- We make the following technical conditions: There is a constant $C_0 > 0$ such that

$$\begin{aligned}&\mathcal{R}^L_x + (C_0 + 2(JT)(\phi)(x))\mathcal{L}_x > 0 \text{ on } T^{1,0}_x X, \forall x \in X,\\ &C_0 + 2(JT)(\phi)(x) > 0, \forall x \in X.\end{aligned} \tag{1}$$

- The conditions (1) always hold if $X := \partial M$ is weakly pseudoconvex.
- The conditions (1) hold for some class of pseudoconcave manifolds.

Theorem (Hsiao-L.-Marinescu, 2022)

Let $M \Subset M'$ be a complex manifold with compact smooth boundary. Let $(L, h) \to M'$ be a positive holomorphic line bundle. Assume that M' admits a holomorphic $\mathbb{R}$-action which can be lifted to L, preserves the boundary ∂M and satisfies transversal property. Under the technical condition as above, L is big on M, i.e., we have

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty.$$

Set up

- Fix a $\mathbb{R}$-invariant Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TM'$.
- Take a $\mathbb{R}$-invariant defining function ρ so that $|d\rho| = 1$ on X.
- $(\cdot|\cdot)_{\overline{M},k}$: inner product on $C^\infty(\overline{M}, L^k)$ induced by $\langle\cdot|\cdot\rangle$ and h^k.
- Let $L^2(M, L^k)$ be the completion of $C^\infty(\overline{M}, L^k)$ under $(\cdot|\cdot)_{\overline{M},k}$.

The operator $-iT$

- $-iT : C^\infty(\overline{M}, L^k) \to C^\infty(\overline{M}, L^k)$.
- We still use the notation $-iT$ for the closure in $L^2(M, L^k)$ of the $-iT$.
- $-iT : \mathrm{Dom}(-iT) \subset L^2(M, L^k) \to L^2(M, L^k)$ is self-adjoint.
- The $\mathrm{Spec}\,(-iT) \subset \mathbb{R}$ is countable, discrete and every element in $\mathrm{Spec}\,(-iT)$ is an eigenvalue of $-iT$.
- For $\alpha \in \mathrm{Spec}(-iT)$, $H^0_{b,\alpha}(X, L^k) := \{u \in H^0_b(X, L^k) : -iTu = \alpha u\}$.
- For $\delta > 0$, $H^0_{b,[\frac{k\delta}{2},\ k\delta]}(X, L^k) := \bigoplus_{\alpha \in \mathrm{Spec}(-iT), \frac{k\delta}{2} \le \alpha \le k\delta} H^0_{b,\alpha}(X, L^k)$.

Theorem (Hsiao-Herrmann-L., Math. Res. Lett., to appear)

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n-1, n \geq 2$ with a transversal and CR $\mathbb{R}$-action. Let $L \to X$ be a positive CR line bundle over X. Assume that the $\mathbb{R}$-action can be lifted to the L. Then

$$\dim H^0_{b,[\frac{k\delta}{2},\ k\delta]}(X, L^k) \gtrsim k^n, k \to \infty.$$

Question: Can the sections in $H^0_{b,[\frac{k\delta}{2},\ k\delta]}(X, L^k)$ be holomorphically extended to M? If so, then

$$\dim H^0(M, L^k) \gtrsim k^n, k \to \infty.$$

Weighted projection

- Fix $\delta > 0$ s.t.
  $\mathcal{R}_x^{L_X} + 2t\mathcal{L}_x > 0, \forall x \in X, t \in [\frac{\delta}{4}, 2\delta]$,
  $t + (JT)(\phi)(x) > 0, \forall x \in X, t \in [\frac{\delta}{4}, 2\delta]$.
- Choose $\tau_\delta(t) \in C_0^\infty(\frac{\delta}{4}, 2\delta)$ s.t. $0 \leq \tau_\delta \leq 1, \tau_\delta \equiv 1, t \in [\frac{\delta}{2}, \delta]$.
- $\tau_{k\delta}(t) := \tau_\delta(\frac{t}{k})$.
- For $\alpha \in \mathrm{Spec}(-iT)$,
  $Q_\alpha : L^2(X, L^k) \to L^2_\alpha(X, L^k)$: orthogonal projection.
- $F_{k\delta,X} : L^2(X, L^k) \to L^2(X, L^k)$: weighted projection.
- $F_{k\delta,X}(u) := \sum_{\alpha \in \mathrm{Spec}(-iT)} \tau_{k\delta}(\alpha) Q_\alpha u, \forall u \in L^2(X, L^k)$.

Sketch of proof of the main result

- $S_k : L^2(X, L^k) \to \mathrm{Ker}(\overline{\partial}_b)$: Szegő projection.
- Weighted Szegő projection:
  $S_{\tau_{k\delta}} := S_k \circ F_{k\delta,X} : L^2(X, L^k) \to \mathrm{Ker}(\overline{\partial}_b)$.
- $P_k^{(0)} : C^\infty(X, L^k) \to C^\infty(\overline{M}, L^k)$: Poisson operator with respect to complex Laplacian $\Box_{f,k}^{(0)} := \overline{\partial}_f^* \overline{\partial}$, i.e.,
- $\overline{\partial}_f^* \overline{\partial} P_k^{(0)} = 0$, $\gamma P_k^{(0)} = \mathrm{id}$, where γ is the restriction operator to the boundary X.
- $\overline{\partial}_f^* \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} = 0$.
- Claim: $\overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} = 0$ when k is large.
- The Claim implies that

$$\mathrm{dim} H^0(M, L^k) \gtrsim k^n, k \to \infty.$$

Sketch of proof of the Claim

- $\overline{\partial}^*_f \overline{\partial} P^{(0)}_k S_{\tau_{k\delta}} = 0.$
- $\gamma \overline{\partial}^*_f \overline{\partial} P^{(1)}_k \gamma \overline{\partial} P^{(0)}_k S_{\tau_{k\delta}} = 0.$

Since $\overline{\partial}_b S_{\tau_{k\delta}} = 0$, one can show that

- $$\begin{aligned} 0 &= 2((\overline{\partial}\rho)\wedge) \circ \gamma \overline{\partial}^*_f \overline{\partial} P^{(0)}_k S_{\tau_{k\delta}} \\ &= 2((\overline{\partial}\rho)\wedge) \gamma \overline{\partial}^*_f P^{(1)}_k \gamma \overline{\partial} P^{(0)}_k S_{\tau_{k\delta}} \\ &= (N_k + iT + Z_0) \gamma \overline{\partial} P^{(0)}_k S_{\tau_{k\delta}} \end{aligned}$$

- $N_k := \gamma(\frac{\partial}{\partial \rho})^* P^{(1)}_k$: Neumann operator.
- Z_0: k-independent zero order differential operator.

Sketch of proof of the Claim

- The Neumann operator N_k satisfies the following uniform estimate:

$$(N_k \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u | \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u)_X \leq C \| \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u \|_X^2 .$$

- $C > 0$ is a constant independent of k.
- $(-iT \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u | \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u)_X \geq \frac{k\delta}{4} \| \gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u \|_X^2 .$
- For k large, $\gamma \overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u = 0, \forall u \in C^\infty(X, L^k)$.
- $\overline{\partial} P_k^{(0)} S_{\tau_{k\delta}} u = 0, \forall u \in C^\infty(X, L^k)$ when k is large.

Thanks a lot!