A pluripotential proof of the uniform Yau–Tian–Donaldson conjecture

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Outline

1. Motivation

Monge–Ampère equations

Classcial setup: let (X,ω) be an n-dimensional compact Kähler manifold. Let $dV = e^f\omega^n$ be a smooth positive volume form on X with

$$\int_X \omega^n = \int_X dV.$$

Put

$$\mathcal{H}_\omega := \left\{\varphi \in C^\infty(X,\mathbb{R}) \middle| \omega_\varphi := \omega + dd^c\varphi > 0\right\}.$$

Question

For $\lambda \in \mathbb{R}$, is there a Kähler potential $\varphi \in \mathcal{H}_\omega$ solving

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} dV?$$

Geometric meaning

By rescaling the Kähler class, may assume that $\lambda \in \{1, 0, -1\}$. Suppose that $\varphi \in \mathcal{H}_\omega$ solves the above equation.

- $\lambda = -1$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = -\omega_\varphi + \eta$$
for some smooth form $\eta \in c_1(X) + [\omega]$.
- $\lambda = 0$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = \eta$$
for some smooth form $\eta \in c_1(X)$.
- $\lambda = 1$: ω_φ satisfies
$$\mathrm{Ric}(\omega_\varphi) = \omega_\varphi + \eta$$
for some smooth form $\eta \in c_1(X) - [\omega]$.

Note: all these metrics are twisted Kähler–Eisntein metrics.

Solvability

For the equation

$$(\omega + dd^c\varphi)^n = e^{-\lambda\varphi} dV$$

- $\lambda = -1$: can always be solved, by Yau and Aubin independently.
- $\lambda = 0$: can always be solved, by Yau's celebrated solution of the Calabi conjecture.
- $\lambda = 1$: There are obstructions related to K-stability/Ding stability.

In what follows we focus on the case of $\lambda = 1$.

Let $\xi \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class (i.e., it contains a Kähler current). Fix a smooth representiative, say $\theta \in \xi$. Let $\mathcal{PSH}(\theta)$ denote the set of all θ-psh functions on X. For any $\varphi \in \mathcal{PSH}(\theta)$, one can define the non-pluripolar Monge–Ampère measure $(\theta + dd^c\varphi)^n$. Then as in the Kähler setting, one can consider the following Monge–Ampère equation (degenerate Kähler–Einstein type equations):

$$(\theta + dd^c\varphi)^n = e^{-\lambda\varphi} dV,$$

where $\lambda \in \mathbb{R}$.

By the work of Berman–Eyssidieux–Guedj–Zeriahi, one can always solve this equation when $\lambda \leq 0$.

Question: what happens when $\lambda > 0$?

More degeneracy on the right hand side

Let χ be a quasi-psh (locally smooth+psh) function on X with analytic singularities, i.e., locally it is of the form

$$\chi = c \log \sum_{i=1}^{r} |f_i|^2 + g$$

for some $c > 0$, $f_i \in \mathcal{O}$ and $g \in C^\infty$. Let ψ be another quasi-psh function on X and consider the following even more degenerate Monge–Ampère equation:

$$(\theta + dd^c\varphi)^n = e^{\chi - \lambda\varphi - \psi}\omega^n.$$

The right hand side may have both zeros and poles, so it is more degenerate! To make sense of this equation, it is necessary to assume the klt condition: $\int_X e^{\chi-\psi}\omega^n < \infty$.
We are able to deal with such degenerate right hand side, thanks to Guan–Zhou's strong openness.

Why do we care?

Typical example: The following degenerate Monge–Ampère equation often appears in the literature:

$$(\theta + dd^c\varphi)^n = e^{-\lambda\varphi} \frac{\prod_{i=1}^{k} ||\sigma_i||^{2a_i}}{\prod_{j=1}^{r} ||s_j||^{2(1-\beta_j)}} \omega^n,$$

which arises naturally when one tries to find Kähler–Einstein metrics on singular Fano varieties with klt singularities.

2. Ding functional

Let $\xi = \{\theta\} \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class. Put

$$V_\theta := \sup\{\varphi \in \mathcal{PSH}(\theta) | \varphi \leq 0\}.$$

Any $\varphi \in \mathcal{PSH}(\theta)$ with $|V_\theta - \varphi| = O(1)$ is said to have minimal singularities. For those φ, its Monge–Ampère energy $E_\theta(\varphi)$ is defined by

$$E_\theta(\varphi) := \frac{1}{(n+1)\,\mathrm{vol}(\xi)} \sum_{i=0}^{n} \int_X (\varphi - V_\theta)(\theta + dd^c\varphi)^i \wedge (\theta + dd^c V_\theta)^{n-1},$$

where

$$\mathrm{vol}(\xi) := \int_X (\theta + dd^c V_\theta)^n$$

is called the volume of ξ.

The space $\mathcal{E}^1(\theta)$

For any $\varphi \in \mathcal{PSH}(\theta)$, note that $\varphi_j := \max\{V_\theta - j, \varphi\}$ decreases to φ pointwise, along which, $E_\theta(\varphi_j)$ decreases as well. Put

$$E_\theta(\varphi) := \lim_{j \to +\infty} E_\theta(\varphi_j).$$

Define

$$\mathcal{E}^1(\theta) := \{\varphi \in \mathcal{PSH}(\theta) | E_\theta(\varphi) > -\infty\}.$$

This is called the finite energy space. For any $u, v \in \mathcal{E}^1(\theta)$, the d_1-distance between them is given by

$$d_1(u, v) := E_\theta(u) + E_\theta(v) - E_\theta(P_\theta(u, v)),$$

where $P_\theta(u, v) = \sup\{\varphi \in \mathcal{PSH}(\theta) : \varphi \le \min\{u, v\}\}$. Note that $(\mathcal{E}^1(\theta), d_1)$ is a complete metric length space (by Darvas–Di Nezza–Lu).

Ding functional

Let

$$\mu := e^{\chi - \psi}\omega^n.$$

be the measure on the right hand side. It is L^p density for some $p > 1$ by Guan–Zhou's strong openness.
Consider the following functional (for $\lambda > 0$)

$$D_\mu^\lambda(\varphi) := -\frac{1}{\lambda}\log\int_X e^{-\lambda\varphi}d\mu - E_\theta(\varphi), \varphi \in \mathcal{E}^1(\theta).$$

Here λ should satisfy the integrability condition: $\int_X e^{-\lambda V_\theta}d\mu < +\infty$.
This is called Ding functional (first introduced by Prof. W.Y. Ding in the 80's). If φ is a critical point, then up to a constant it holds that (by Berman–Boucksom–Guedj–Zariahi's variational principle)

$$(\theta + dd^c\varphi)^n = e^{-\lambda\varphi}\mu.$$

Properness

Definition

We say D_μ^λ is proper if there exists $\varepsilon > 0$ and $C > 0$ such that

$$D_\mu^\lambda(\varphi) \geq \varepsilon(\sup\varphi - E_\theta(\varphi)) - C \text{ for any } \varphi \in \mathcal{E}_\omega^1.$$

Remark: one should view $\sup\varphi - E(\varphi)$ as a distance function on $\mathcal{E}^1(\theta)$. In fact, $\sup\varphi - E_\theta(\varphi) = d_1(V_\theta, \varphi - \sup\varphi)$.

Theorem (BBGZ's variational principle)

If D_μ^λ is proper, then D_μ^λ admits a minimizer $\varphi \in \mathcal{E}^1(\theta)$ which solves

$$(\theta + dd^c\varphi)^n = e^{-\lambda\varphi}\mu.$$

How do you get the properness?

In general D^{λ}_{μ} is not proper, as there are obstructions to the solvability of the equation. In the spirit of Yau–Tian–Donaldson conjecture, we need to find suitable algebro-geometric condition to guarantee the properness.

3. Valuative delta invariant

Valuative functions

Let E be a prime divisor over X: there exists a bimeromorphic modification $Y \xrightarrow{\pi} X$ such that E is a smooth hypersurface in Y. Put

$$A_X(E) := 1 + \operatorname{ord}_E(K_Y - \pi^* K_X),$$

$$A_\mu(E) := A_X(E) + \nu(\chi, E) - \nu(\psi, E).$$

The klt condition ensures that $A_\mu(E) > 0$. Moreover, let

$$\tau_\xi(E) := \sup\{x > 0 | \pi^*\xi - x\{E\} \text{ is big}\}$$

and

$$S_\xi(E) := \frac{1}{\operatorname{vol}(\xi)} \int_0^{\tau_\xi(E)} \operatorname{vol}(\pi^*\xi - x\{E\}) dx.$$

Here $\operatorname{vol}(\pi^*\xi - x\{E\})$ is understood in the pluripotential sense, which generalizes the notion of volume in algebraic geometry to the transcendental setting (cf. Boucksom).

Valuative delta invariant

Following Fujita–Odaka and Blum–Jonsson, we define a valuative δ-invariant by

$$\delta_\mu(\xi) := \inf_E \frac{A_\mu(E)}{S_\xi(E)}.$$

When $\chi = \psi = 0$, $\xi = c_1(-K_X)$ is ample (Fano case), this is the stability threshold characterizing K-stability, introduced by Fujita–Odaka. When $\xi = c_1(L)$ for an ample line bundle L, $\delta(L)$ characterizes Ding stability.

Conjecture

Conjecture

One has $\delta_\mu(\xi) = \sup\{\lambda > 0 : D_\mu^\lambda \text{ is proper}\}$.

Theorem (Z, 2021, to appear in JEMS)

When ξ is an ample class and $\chi = 0$, the above conjecture is true.

The above result is proved using quantization techniques. To treat general big classes, new ideas are required.
But we do have some partial progress, based on the recent joint work with T. Darvas, which also gives a completely new proof of the classical (uniform) YTD conjecture of Li–Tian–Wang in the log Fano case.

Main result

Theorem (Darvas-Z, 2022)

One has

$$\delta_\mu(\xi) = \sup \left\{ \lambda > 0 : \liminf_{t\to\infty} \frac{D_\mu^\lambda(u_t)}{t} \geq 0 \textit{ for any subgeodesic ray } u_t \in \mathcal{E}^1(\theta) \right\}.$$

If we know that Ding functional is convex on $\mathcal{E}^1(\theta)$, then the above result will imply the previous conjecture. Unfortunately, we do not know whether Ding functional is convex or not in such general setting.
But in practice, when searching for KE metrics on (log) Fano pairs, Ding funcitonal is indeed convex (by Berndtsson)! So our result implies the classical YTD theorems in the log Fano setting.

Corollary (Li–Tian–Wang)

Let (X, Δ) be a log Fano variety with klt singularities and assume that it is uniformly K-stable. Then it admits a Kähler–Einstein metric.

A YTD type result for big classes

Fix a smooth repersentative $\eta \in c_1(X) - \xi$. Assume that $\{\eta\}$ is a pseudoeffective class (i.e., it contains a semi-positive $(1,1)$-current.) Let $\chi = f \in C^\infty$ be such that $\mathrm{Ric}(\omega) = \theta + \eta + dd^c f$ and let $\psi \in \mathcal{PSH}(\eta)$ be such that

$$\int_X e^{-V_\theta - \psi} \omega^n < \infty.$$

This is klt condition ensuring that $-K_X$, as a big line bundle, admits a singular Hermitian metric with positive curvature current and trivial multiplier ideal sheaf. By a result of Berndtsson–Paun, the Ding functional is also convex in this context.

Theorem (Darvas-Z, 2022)

In the above setting, if $\delta_\mu(\xi) > 1$, then there exists $u \in \mathcal{E}^1(\theta)$ such that $(\theta + dd^c u)^n = e^{f-u-\psi}\omega^n$.

Corollary (Darvas-Z, 2022)

Let X be a compact Kähler manifold such that $-K_X$ is big and $\delta(-K_X) > 1$, then there exists a Kähler–Einstein current in $c_1(X)$.

4. The proof: Pluripotential approach

Test curve

Given a subgeodesic ray $(0,\infty) \ni t \mapsto u_t \in \mathcal{E}^1(\theta)$, consider the following Legendre transform

$$\hat{u}_\tau := \inf_{t>0}\{u_t - t\tau\}, \ \tau \in \mathbb{R}.$$

Note that $\hat{u}_\tau \in \mathcal{PSH}(\theta)$ by Kiselman's minimum principle. This is called a test curve by Ross-Witt Nyström. This family of θ-psh function satisfies several properties:

1. $\tau \mapsto \hat{u}_\tau(x)$ is decreasing, concave and usc for any fixed $x \in X$.
2. $\hat{u}_\tau \equiv -\infty$ for $\tau \gg 0$.
3. $\hat{u}_\tau \nearrow V_\theta$ a.e. as $\tau \searrow -\infty$.

Theorem (Darvas-Z, 2022)

For any subgeodesic ray $u_t \in \mathcal{E}^1(\theta)$ one has

$$\liminf_{t\to\infty} \frac{D_\mu^\lambda(u_t)}{t} = \sup\{\tau \in \mathbb{R} : \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty\} - \lim_{t\to\infty} \frac{E_\theta(u_t)}{t}.$$

The limit $\lim_{t\to\infty} \frac{E_\theta(u_t)}{t}$ exists as $E_\theta(\cdot)$ is convex along subgeodesics. So we put

$$E_\theta\{u_t\} := \lim_{t\to\infty} \frac{E_\theta(u_t)}{t},$$

and call it the radial Monge–Ampère energy (or the barycenter of $\{\hat{u}_\tau\}$). Similarly, let

$$D_\mu^\lambda\{u_t\} := \liminf_{t\to\infty} \frac{D_\mu^\lambda(u_t)}{t}$$

be the radial Ding functional. Thus we have that

$$D_\mu^\lambda\{u_t\} = \sup\{\tau \in \mathbb{R} : \int_X e^{-\lambda \hat{u}_\tau} d\mu < \infty\} - E_\theta\{u_t\}.$$

Analytic meaning of delta invariant

For any quasi-psh function u on X, let

$$c_\mu[u] = \sup\{\lambda > 0, \int_X e^{-\lambda u} d\mu < \infty\}.$$

Note that $c_\mu[\cdot]$ must be positive by Guan–Zhou's strong openness theorem. Moreover, combining Guan–Zhou and Boucksom–Favre–Jonsson, one has

$$c_\mu[u] := \inf_E \frac{A_\mu(E)}{\nu(u, E)}.$$

Theorem (Darvas-Z, 2022)

One has

$$\delta_\mu(\xi) = \inf_{\{u_t\}} c_\mu[\hat{u}_{E_\theta\{u_t\}}],$$

where the inf is over all subgodesic ray $\{u_t\}$ in $\mathcal{E}^1(\theta)$.

One can finish the proof of our main result by combining this result with the previous radial formula.

A key inequality

To prove the previous result, a crucial observation is the following inequality.

Proposition

For any subgodesic ray $\{u_t\}$ in $\mathcal{E}^1(\theta)$ and any prime divisor E over X, one has

$$\nu(\hat{u}_{E_\theta\{u_t\}}, E) \leq S_\xi(E),$$

and the equality holds when $\{u_t\}$ is induced by E

This implies that

$$\inf_E \frac{A_\mu(E)}{S_\xi(E)} = \inf_{\{u_t\}} c_\mu[\hat{u}_{E_\theta\{u_t\}}],$$

as desired.

Remark: During the course of our proof, Guan–Zhou's strong openness play dominant role and is used repeatedly.

Thanks for your attention!