Linear isometric invariants of bounded domains and their plurisubharmonic variation

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2022年全国多复变学术年会
上海大学，2022年8月18日

Based on the following works:

(1) F. Deng, Z. Wang, L. Zhang, X. Zhou: *Linear invariants of complex manifolds and their plurisubharmonic variations*, J. Funct. Anal. 279 (2020), no.1, 108514, arXiv:1901.08920.

(2) F. Deng, J. Ning, Z. Wang, X. Zhou: *Linear isometric invariants of bounded domains*, preprint (2022).

Background and motivation

X_1, X_2 are Stein manifolds, then:

$$X_1 \simeq X_2$$

$$\Longleftrightarrow$$

$$\mathcal{O}(X_1) \simeq \mathcal{O}(X_2) \text{ as } \mathbb{C}\text{-algebras with unit.}$$

Background and motivation

Functority property: Given $T : \mathcal{O}(X_1) \simeq \mathcal{O}(X_2)$, can construct $f : X_1 \simeq X_2$:

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma_1:\ bijection} & Spm(\mathcal{O}(X_1)) \\ {\scriptstyle f:=\sigma_2^{-1}\circ T_*\circ\sigma_1}\downarrow & & \downarrow{\scriptstyle T_*} \\ X_2 & \xrightarrow{\sigma_2:\ bijection} & Spm(\mathcal{O}(X_1)) \end{array}$$

- Spm=\{finitely generated maximal ideals\}
- $\sigma_1(z) = \{f \in \mathcal{O}(X_1) | f(z) = 0\}$
- Bijection of σ_i is proved by Cartan A, B.

Background and motivation

Similar picture in algebraic geometry: X_1, X_2 are projective manifolds of general type (canonical bundle big). Let

$$K(X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

be the canonical ring. MMP:

X_1, X_2 are birational

$\Longleftrightarrow$

$K(X_1) \simeq K(X_2)$ as graded $\mathbb{C}$-algebras.

Background and motivation

- In the above examples, product structure on the algebraic objects are very important.
- On the other hand, Banach originated the study on characterization of measure spaces via linear and metric structures of their L^p spaces (without product structure).
- Royden's work on compact Riemann surfaces: X, Y compact R.S. of genus ≥ 2, then

$$X \simeq Y$$

$$\Longleftrightarrow$$

$$H^0(X, 2K_X) \simeq H^0(Y, 2K_Y) \text{ as Banach spaces.}$$

- Markovic's generalization to noncompact Riemann surfaces, and Chi-Yau's generalization to higher dimensional projective manifolds.
- This lecture aims to present a generalization to bounded domains.

Linear invariants of bounded domains

Let $D \subset \mathbb{C}^n$ be bounded domain and $p > 0$, we define:

$$A^p(D) = \{\phi \in \mathcal{O}(D) | \|\phi\|_p := \left(\int_D |\phi|^p\right)^{1/p} < \infty\},$$

$$B_{D,p}(z) := \sup_{\phi \in A^p(D)} \frac{|\phi(z)|^2}{\|\phi\|_p^2}.$$

Definition

$B_{D,p}(z)$ is called the p-Bergman kernel of D. When $p = 2$, it is the ordinary Bergman kernel.

From now on, we always assume $p > 0$ is not an integer.

Basic notions

Definition

$B_{D,p}(z)$ is called exhaustive if $\{z \in D | B_{D,p}(z) \leq c\} \Subset D$ for any $c > 0$.

Definition

D is called hyperconvex if $\exists$ a p.s.h function $\rho : D \to [-\infty, 0)$ s.t. $\forall c < 0$ the set $\{z \in D | \rho(z) \leq c\} \Subset D$.

Definition

That $T : A^p(D_1) \to A^p(D_2), \phi \mapsto T\phi$ is a linear isometry means that T is a linear isomorphism and $\|T\phi\|_p = \|\phi\|_p$ for all $\phi \in A^p(D_1)$.

Linear invariants of bounded domains

Theorem (D-Wang-Zhang-Zhou)

Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be bounded hyperconvex domains. Suppose $\exists\ p > 0$ such that

(1) $\exists\ T : A^p(D_1) \to A^p(D_2)$ *linear isometry,*

(2) *the p-Bergman kernels of D_1 and D_2 are exhaustive.*

Then $m = n$ and $D_1 \cong D_2$

- V. Markovic (2003): for the case $D_1, D_2 \subset \mathbb{C}$ and $p = 1$, motivated by Teichmüller theory.
- Recently, Inayama get a relative version of it.

Remark: $A^p(X)$ can be defined intrinsically for a complex manifold X if p is of the form $\frac{2}{m}$, $m \in \mathbb{N}$.

A^p-completeness

We discuss and improvement of the above result.
We first relax the condition of p-Bergman kernel exhaustion.

Definition

A bounded domain $D \subset \mathbb{C}^n$ is A^p-complete if $\nexists\ \tilde{D} \subset \mathbb{C}^n$ with $D \subsetneqq \tilde{D}$ such that the restriction map $i : A^p(\tilde{D}) \to A^p(D)$ is a linear isometry.

For example, D is A^p-complete if:

(1) the p-Bergman kernel of D is exhaustive; or

(2) $\mathring{\overline{D}} = D$, namely, the interior of the closure of D is D itself.

A^p-completeness

The importance of A^p-completeness is encoded in the following

Theorem (D-Ning-Wang-Zhou)

Assume $D_1, D_2 \Subset \mathbb{C}^n$ are A^p-complete, and $\exists$ a linear isometry $T : A^p(D_1) \to A^p(D_2)$. Then $\exists$ hypersurfaces $A_1 \subset D_1, A_2 \subset D_2$ (may be empty) and a biholomorphic map

$$F : D_1 \backslash A_1 \to D_2 \backslash A_2$$

with

$$F_1(A_1) \subset \partial D_2, \ F_2(A_2) \subset \partial D_1.$$

We can glue D_1 and D_2 via F to get a complex manifold.

Boundary blow down type

We second relax the condition of hyperconvexity of D. Motivated by the above theorem, we propose the following

Definition

$D \subset \mathbb{C}^n$ is of boundary blow down type, if there exists a complex manifold M, a (nonempty) hypersurface $A \subset M$, $h \in \mathcal{M}(M) \cap \mathcal{O}(M \setminus A)$, and a holomorphic map $\sigma : M \to \mathbb{C}^n$, such that

(i) $\sigma(M \backslash A) = D$ *and* $\sigma(A) \subset \partial D$,

(ii) $\sigma|_{M \backslash A} : M \backslash A \to D$ *is a biholomorphic map,*

(iii) $h^{-1}(\infty) = A$.

Boundary blow down type

Some examples of domains that are NOT of boundary blow down type:

(i) Runge domains,

(ii) hyperconvex domains

(iii) $D \backslash K$, where D is Runge or hyperconvex and $K \subset D$ compact.

Remark:
(iii) above makes us able to handle some domains that are not pseudoconvex.

Main result

Theorem (D-Ning-Wang-Zhou)

Assume $D_1, D_2 \Subset \mathbb{C}^n$ *are* A^p*-complete and are not of boundary blow down type, and* $T : A^p(D_1) \to A^p(D_2)$ *is a linear isometry for some* $p > 0$*. Then* $\exists!$ *biholomorphic map* $F : D_1 \to D_2$ *s.t.*

$$T\phi(F(z))J_F(z)^{2/p} = \lambda\phi(z),\ \forall\phi \in A^p(D_1), z \in D_1,$$

where $\lambda \in \mathbb{C},\ |\lambda| = 1$*.*

Application to domains with certain boundary regularity

Corollary

Assume $D_1, D_2 \Subset \mathbb{C}^n$ *are pseudoconvex domains with Hölder boundary. If there exists a linear isometry between* $A^p(D_1)$ *and* $A^p(D_2)$ *for some* $p > 0$*, then* $D_1 \cong D_2$*.*

A conjecture

For $p \in (0,2)$, we believe that the condition of not being BBDT in the above theorem is not necessary.

Conjecture

Let $D_1, D_2 \Subset \mathbb{C}^n$ be A^p-complete for some $p \in (0,2)$. If there is a linear isometry between $A^p(D_1)$ and $A^p(D_2)$, then $D_1 \cong D_2$.

A counter-example can be constructed if $p > 2$.

How to construct the map $F : D_1 \cong D_2$?

Maximal ideals replaced by hypersurfaces:

$$\begin{array}{ccc} D_1 & \xrightarrow{\sigma_1} & \mathbb{P}(A^p(D_1)^*) \supseteq \sigma_1(D_1) \\ \Big\downarrow F:=\sigma_2^{-1}\circ T_*\circ\sigma_1 & & \Big\downarrow T_* \\ D_2 & \xrightarrow{\sigma_2} & \mathbb{P}(A^p(D_2)^*) \supseteq \sigma_2(D_2) \end{array}$$

- $P(A^p(D_1)^*) := \{\text{hyperplanes in } A^p(D_1)\}$.
- $\sigma_1(z) = \{\phi \in A^p(D_1) | \phi(z) = 0\} \in P(A^p(D_1)^*)$.
- need show that $T_*\sigma_1(D_1) = \sigma_2(D_2)$, and F is biholomorphic.

Further study

Generalize the methods and results to complex manifolds equipped with Hermitian holomorphic vector bundles and develop some relative version.

Plurisubharmonic variation of linear invariants

Consider pseuconvex domain $\Omega \subset \mathbb{C}_t^n \times \mathbb{C}_z^m$ and let $B = p(\Omega) \subset \mathbb{C}^n$, where $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ is the natural projection. Let $\Omega_t = p^{-1}(t)$ and $E_t = A^p(\Omega_t)$ for $t \in B$. Then

$$E := \bigsqcup_{t \in B} E_t \to B$$

can be roughly seen as a vector bundle over B with a (singular) Finsler metric.

Plurisubharmonic variation of linear invariants

Theorem (D-Wang-Zhang-Zhou)

The curvature of E is semipositive in the sense of Griffiths for $p \in (0, 2]$.

Proof based on optimal L^2-extension theory developed in recent years by Guan, Zhou, and Zhu.

Similar results also holds for more general families of complex manifolds.

谢谢!