Deformations of Dirac operators and applications

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Abstrac

- Dirac operators as well as their deformations have played important roles in many problems in geometry and topology. We will survey some of these applications in this talk.

Dirac operator : the origins

- 1846 Hamilton :

$$-\left(\frac{i\mathrm{d}}{\mathrm{d}x}+\frac{j\mathrm{d}}{\mathrm{d}y}+\frac{k\mathrm{d}}{\mathrm{d}z}\right)^2=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2$$

- 1928 Dirac

$$\left(\sum_{i=0}^{3}\gamma_i\frac{\partial}{\partial x^i}\right)^2=-\left(\frac{\partial}{\partial x^0}\right)^2+\sum_{i=1}^{3}\left(\frac{\partial}{\partial x^i}\right)^2$$

- γ_i's verify the (later) so called Clifford relations
- 1960s Atiyah-Singer on spin manifolds

Dirac operators on spin manifolds

- (M^{2n}, g^{TM}) a closed Riemannian spin manifold, with Levi-Civita connection ∇^{TM}, $R^{TM} = (\nabla^{TM})^2$
- $S(TM) = S_+(TM) \oplus S_-(TM)$ Hermitian bundle of spinors, carry Hermitian connection $\nabla^{S(TM)}$
- (E, g^E) Hermtian vector bundle on M, with Hermitian connection ∇^E, with curvature $R^E = (\nabla^E)^2$
- Atiyah-Singer's Dirac operator :

$$D^E = \sum_{i=1}^{2n} c(e_i) \nabla_{e_i}^{S(TM)\otimes E} : \Gamma(S(TM)\otimes E) \longrightarrow \Gamma(S(TM)\otimes E),$$

$$D_\pm^E = D^E|_{\Gamma(S_\pm(TM)\otimes E)}, \quad \operatorname{ind}(D_+^E) = \ker(D_+^E) - \ker(D_-^E)$$

- **Atiyah-Singer index theorem (1963)**

$$\operatorname{ind}\left(D_+^E\right) = \left\langle \widehat{A}(TM)\operatorname{ch}(E), [M] \right\rangle$$

Dirac operators on spin manifolds

- In Chern-Weil form :

$$\operatorname{ind}\left(D_+^E\right) = \int_M \det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right)\operatorname{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]$$

- Spin condition essential : $\widehat{A}(\mathbf{C}P^2) = -\frac{1}{8}$

- Early application : Lichnerowicz formula :

$$D^2 = -\Delta + \frac{k^{TM}}{4},$$

where k^{TM} is the scalar curvature of g^{TM}.

- **Lichnerowicz (1963)** If $k^{TM} > 0$, then $\widehat{A}(M) = 0$.

Geometric operators as Dirac operators

- Locally, every manifold is spin
- Canonical geometric operators locally can be seen as Dirac operators

- Example 1 : de Rham-Hodge operator on a Riemannian manifold M

$$\mathrm{d} + \mathrm{d}^* : \Omega^*(M) \longrightarrow \Omega^*(M),$$

where $\Omega^*(M) = \Gamma(\Lambda^*(T^*M))$.

- **Gauss-Bonnet-Chern theorem (1940s)**

$$\chi(M) = \int_M \mathrm{Pf}\left(\frac{R^{TM}}{2\pi}\right)$$

Geometric operators as Dirac operators

- Example 2 : Dolbeault operator for a holomorphic vector bundle L on a Kähler manifold M

$$\sqrt{2}\left(\overline{\partial}^L + \left(\overline{\partial}^L\right)^*\right) : \Omega^{0,*}(M,L) \longrightarrow \Omega^{0,*}(M,L)$$

- **Riemann-Roch-Hirzebruch theorem (1950s)**

$$\sum_{i=0}^{n} (-1)^i \dim H^{0,i}(M,L) = \langle \mathrm{Td}(TM)\mathrm{ch}(L), [M] \rangle$$

- In many applications, **deformations** of Dirac operators (geometric operators) play important roles, we will indicate some examples in this talk

- Basic principle behind : $\mathrm{ind}(F+K) = \mathrm{index}(F)$

Early example : Atiyah's proof of the Hopf vanishing theorem

- **Hopf vanishing theorem :** If V is a nowhere zero vector field on a closed manifold M, then $\chi(M) = 0$.

- Take a metric g^{TM} on TM.

- Let $\mathrm{d}^* : \Omega^*(M) \to \Omega^*(M)$ be the formal adjoint of the exterior differential $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$.

- Recall that by the Hodge theorem,

$$\chi(M) = \mathrm{ind}\left((\mathrm{d} + \mathrm{d}^*)_+\right) : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M).$$

Clifford actions on $\Omega^*(M) = \Gamma(\Lambda^*(T^*M))$

- Given g^{TM}, two standard Clifford actions on $\Omega^*(M)$:
- For any $X \in TM$, let $X^* \in T^*M$ be dual to X via g^{TM}. Set

$$c(X) = X^* \wedge -i_X, \quad \widehat{c}(X) = X^* + i_X.$$

- For any X, $Y \in TM$, Clifford relations :

$$c(X)c(Y) + c(Y)c(X) = -2\langle X, Y\rangle_{g^{TM}},$$
$$\widehat{c}(X)\widehat{c}(Y) + \widehat{c}(Y)\widehat{c}(X) = 2\langle X, Y\rangle_{g^{TM}},$$
$$c(X)\widehat{c}(Y) + \widehat{c}(Y)c(X) = 0.$$

- $$\mathrm{d} + \mathrm{d}^* = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)},$$

  where $\{e_i\}_{i=1}^{\dim M}$ is a (local) orthonormal basis of (TM, g^{TM}), $\nabla^{\Lambda^*(T^*M)}$ is induced from the Levi-Civita connection ∇^{TM} of (TM, g^{TM}).

Atiyah's proof of the Hopf vanishing theorem

- Recall that $V \in \Gamma(TM)$ with $\mathrm{zero}(V) = \emptyset$.
- Take g^{TM} such that $|V|_{g^{TM}} = 1$
- Following Atiyah (1970), one has

$$\chi(M) = \mathrm{ind}\left(\mathrm{d} + \mathrm{d}^* : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)\right)$$

$$= \mathrm{ind}\left(\widehat{c}(V)\,(\mathrm{d} + \mathrm{d}^*)\,\widehat{c}(V) : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)\right).$$

- Now by the Clifford relations,

$$\widehat{c}(V)\,(\mathrm{d} + \mathrm{d}^*)\,\widehat{c}(V) = -\,(\mathrm{d} + \mathrm{d}^*) + \widehat{c}(V) \sum_{i=1}^{\dim M} c(e_i)\widehat{c}\left(\nabla^{TM}_{e_i} V\right),$$

which implies

$$\mathrm{ind}\left(\widehat{c}(V)\,(\mathrm{d} + \mathrm{d}^*)\,\widehat{c}(V) : \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{even}}(M)\right) = -\chi(M).$$

- Thus, $\chi(M) = -\chi(M)$ from which one gets $\chi(M) = 0$.

Witten's analytic proof of Morse inequalities

- Witten (1982) : for any $f \in C^\infty(M)$ and $T \in \mathbf{R}$,

$$\mathrm{d}_{Tf} = e^{-Tf}\mathrm{d}e^{Tf}.$$

- Let $\mathrm{d}^*_{Tf} = e^{Tf}\mathrm{d}^* e^{-Tf}$ be the formal adjoint of d_{Tf}
- By considering the deformed Laplacian

$$\Box_{Tf} = \left(\mathrm{d}_{Tf} + \mathrm{d}^*_{Tf}\right)^2 = \mathrm{d}_{Tf}\mathrm{d}^*_{Tf} + \mathrm{d}^*_{Tf}\mathrm{d}_{Tf},$$

Witten gets an analytic proof of the Morse inequalities.

- Witten's proof is very influential. On the non-linear side, it motivates Floer (1988) to introduce his homology. On the linear side, Bismut-Zhang (1992) make use of the Witten deformation to give a purely analytic proof of the Cheeger-Müller theorem (1978) concerning the Ray-Singer analytic torsion and the Reidemeister torsion.

Witten's analytic proof of the Hopf index formula

- One has for the previous Witten deformation that

$$\mathrm{d}_{Tf} + \mathrm{d}_{Tf}^* = \mathrm{d} + \mathrm{d}^* + T\widehat{c}(\nabla f).$$

- Replace ∇f by any $V \in \Gamma(TM)$, one considers

$$D_T = \mathrm{d} + \mathrm{d}^* + T\widehat{c}(V) = \sum_{i=1}^{\dim M} c(e_i)\nabla_{e_i}^{\Lambda^*(T^*M)} + T\widehat{c}(V),$$

from which one gets

$$D_T^2 = (\mathrm{d} + \mathrm{d}^*)^2 + T\sum_{i=1}^{\dim M} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM} V\right) + T^2|V|^2.$$

- If $\mathrm{zero}(V) = \emptyset$, then when $T > 0$ is large enough, $D_T^2 > 0$ is invertible, from which we get another proof of the Hopf vanishing theorem :

$$\chi(M) = \mathrm{ind}\left(D_T : \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)\right) = 0.$$

Witten's analytic proof of the Hopf index formula

- Now we allow $V \in \Gamma(TM)$ to have non-degenerate isolated zeroes. For any $p \in \mathrm{zero}(V)$, take a sufficiently small open neighborhood U_p of p, then when $T >> 0$,

$$D_T^2 = (\mathrm{d}+\mathrm{d}^*)^2 + T\sum_{i=1}^{\dim M} c(e_i)\widehat{c}\left(\nabla_{e_i}^{TM}V\right) + T^2|V|^2 >> 0$$

  on $M \setminus \cup_{p\in \mathrm{zero}(V)} U_p$.

- Thus the study of $\ker(D_T)$ "localizes" to each U_p when $T >> 0$.

- The harmonic oscillator comes into the picture!

- Harmonic oscillator on $\mathbf{R}$: $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 - 1$

Witten's analytic proof of the Hopf index formula

- Near each $p \in \mathrm{zero}(V)$, one considers the harmonic oscillator on $(U_p, \Omega^*(M)|_{U_p}) \simeq (\mathbf{R}^{\dim M}, \Omega^*(\mathbf{R}^{\dim M}))$ to get
- **Poincaré-Hopf index formula :**

$$\chi(M) = \sum_{p \in \mathrm{zero}(V)} \mathrm{ind}_V(p).$$

From Witten to Bismut-Lebeau

- If zero(V) is non-degenerate in the sense of Bott where the set of critical points consists of submanifolds instead of points, then one can get a generalized Hopf formula.
- Harmonic oscillator analysis along the normal directions to submanifolds

- Bismut-Lebeau (1991) : far reaching generalizations to the problem on Quillen metrics for complex immersions
- Essential for Gillet-Soulé's arithmetic Riemann-Roch

- Wide range of applications : the systematic "analytic localization techniques" developed by Bismut-Lebeau

The Guillemin-Sternberg geometric quantization

- Let L be a holomorphic Hermitian line bundle over a Kähler manifold (M, ω), admitting a Hermitian connection ∇^L such that
$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega$$
- G compact Lie group, with Lie algebra $\mathbf{g}$.
- Assume there is a holomorphic Hamiltonian action of G on (M, ω) : there is a moment map
$$\mu : M \longrightarrow \mathbf{g}^*$$
such that for any $X \in \mathbf{g}$,
$$i_{X_G}\omega = \mathrm{d}\left\langle \mu, X \right\rangle,$$
where X_G is the Killing vector field generated by X.

- We assume G also acts holomorphically on L and preserves g^L, ∇^L

The Guillemin-Sternberg geometric quantization

- Assume $0 \in \mathbf{g}^*$ is a regular value of $\mu : M \to \mathbf{g}^*$
- $M_G = \mu^{-1}(0)/G$ is an orbifold, called the symplectic reduction of the G-action.
  For simplicity we assume its smooth
- We get a line bundle (L_G, ∇^{L^G}) over the induced Kähler manifold (M_G, ω_G) with $\frac{\sqrt{-1}}{2\pi}\left(\nabla^{L^G}\right)^2 = \omega_G$
- **Guillemin-Sternberg (1982)**

$$\dim H^{0,0}(M, L)^G = \dim H^{0,0}(M_G, L_G)$$

- "Quantization commutes with reduction"

The Guillemin-Sternberg geometric quantization

- Guillemin-Sternberg geometric quantization conjecture :

$$\sum_{i=0}^{n}(-1)^i \dim H^{0,i}(M,L)^G = \sum_{i=0}^{n}(-1)^i \dim H^{0,i}(M_G,L_G)$$

- Natural symplectic setting, proved by Meirenken (1998)

- **Youliang Tian - Zhang (1998)** : analytic proof

$$D_T^L = \sqrt{2}\left(e^{-T|\mu|^2}\overline{\partial}^L e^{T|\mu|^2} + e^{T|\mu|^2}\left(\overline{\partial}^L\right)^* e^{-T|\mu|^2}\right)$$

(holomorphic analogue of the Witten deformation)

- **Braverman-Teleman-Zhang (1999)** One has for any $i \geq 0$ and $p \geq 0$,

$$\dim H^{0,i}\left(M,L^p\right)^G = \dim H^{0,i}\left(M_G,L_G^p\right)$$

Abstract algebraic structure

- $\xi = \xi_+ \oplus \xi_-$ and $\mathbf{Z}_2$-graded Hermitian vector bundle carrying a Hermitian connection $\nabla^\xi = \nabla^{\xi_+} + \nabla^{\xi_-}$, over an even dimensional spin manifold (M, g^{TM}).
- V a self-adjoint odd endomorphism of ξ (i.e., exchanges $\xi_\pm$)
- $D^\xi : \Gamma(S(TM)\widehat{\otimes}\xi) \to \Gamma(S(TM)\widehat{\otimes}\xi)$ is self-adjoint, with

$$D_+^\xi : \Gamma\left(S_+(TM)\widehat{\otimes}\xi_+ \oplus S_-(TM)\widehat{\otimes}\xi_-\right)$$

$$\longrightarrow \Gamma\left(S_-(TM)\widehat{\otimes}\xi_+ \oplus S_+(TM)\widehat{\otimes}\xi_-\right)$$

- For any $T \in \mathbf{R}$, one considers the deformation

$$D_T^\xi = D^\xi + TV$$

with

$$\left(D_T^\xi\right)^2 = \left(D^\xi\right)^2 + T\left[D^\xi, V\right] + T^2V^2$$

A relative index formula

- Now assume (M, g^{TM}) is complete, and that there is a compact subset $K \subset M$ such that
$$|V|^2 \geq \delta > 0 \quad \text{on} \quad M \setminus K$$
Moreover, we assume on $M \setminus K$ that
$$\left[\nabla^{\xi}, V\right] = 0$$
- Under the above assumptions, one has on $M \setminus K$ that
$$\left(D_T^{\xi}\right)^2 = \left(D^{\xi}\right)^2 + T^2 V^2 \geq T^2 \delta > 0$$
which implies that $\dim(\ker D_T^{\xi}) < +\infty$ for any $T > 0$.

- **A relative index theorem (2021)** For any $T > 0$,
$$\operatorname{ind}\left(D_{T,+}^{\xi}\right) = \left\langle \widehat{A}(TM)\left(\operatorname{ch}\left(\xi_+\right) - \operatorname{ch}\left(\xi_-\right)\right), [M]\right\rangle$$

The Gromov-Lawson relative index theorem.

- Originally, Gromov-Lawson assume that $k^{TM} \geq \widetilde{\delta} > 0$ outside a compact subset $K \subset M$, and that $\xi_\pm$ are trivial bundles on $M \setminus K$.
- By the Lichnerowicz formula, one has

$$\left(D_T^\xi\right)^2 = -\Delta^\xi + \frac{k^{TM}}{4} + T^2V^2 \geq \frac{\widetilde{\delta}}{4} + T^2\delta \quad \text{on} \quad M \setminus K$$

Thus $\dim(\ker D_T^\xi) < +\infty$ for $T \in \mathbf{R}$. Take $T = 0$, one gets

- **Gromov-Lawson relative index theorem (1983)**

$$\operatorname{ind}\left(D_+^{\xi_+}\right) - \operatorname{ind}\left(D_+^{\xi_-}\right) = \left\langle \widehat{A}(TM)\left(\operatorname{ch}\left(\xi_+\right) - \operatorname{ch}\left(\xi_-\right)\right), [M] \right\rangle.$$

Area enlargeability of Gromov-Lawson

- (M, g^{TM}) a Riemannian manifold of dimension n
- Gromov-Lawson : (M, g^{TM}) area enlargeable if for any $\epsilon > 0$, there is a covering $\pi : \widehat{M}_\epsilon \to M$ (with lifted metric) and a smooth map $f : \widehat{M}_\epsilon \to S^n(1)$, which is constant near infinity (that is, constant outside a compact subset) and of nonzero degree, such that for any $\alpha \in \Omega^2(S^n(1))$, one has $|f^*\alpha| \leq \epsilon|\alpha|$
- If M is compact, then the area enlargeability does not depend on g^{TM}
- Typical examples : T^n. Also if M is closed area enlargeable and N is a closed manifold of the same dimension, then $M\#N$ is area enlargeable

Area enlargeability and positive scalar curvature

- **Gromov-Lawson (1983).** If (M, g^{TM}) is a complete spin area enlargeable manifold, then the scalar curvature k^{TM} of g^{TM} can not have a positive lower bound, i.e.,

$$\inf\left(k^{TM}\right) \leq 0.$$

- **Proof.** Assume n is even.
  Consider the map $f : \widehat{M}_\epsilon \to S^n(1)$.
  Let E be a Hermitian vector bundle on $S^n(1)$ such that

$$\langle \mathrm{ch}(E), [S^n(1)] \rangle \neq 0$$

- Consider the Dirac operator D^{f^*E} on $\widehat{M}_\epsilon$.
- By Lichnerowicz formula

$$\left(D^{f^*E}\right)^2 = -\Delta^{f^*E} + \frac{k^{T\widehat{M}_\epsilon}}{4} + c\left(R^{f^*E}\right)$$

Area enlargeability and positive scalar curvature

- By area enlargeability,

$$c\left(R^{f^*E}\right) = c\left(f^*\left(R^E\right)\right) = O(\epsilon)$$

- If $k^{TM} \geq \delta > 0$, then when $\epsilon > 0$ is small enough, one has $(D^{f^*E})^2 > 0$, which implies $\mathrm{ind}(D_+^{f^*E}) = 0$.
- By the Gromov-Lawson relative index theorem, one gets

$$0 = \mathrm{ind}\left(D_+^{f^*E}\right) - \mathrm{rk}(E)\,\mathrm{ind}\left(D_+\right)$$

$$= \left\langle \widehat{A}\left(T\widehat{M_\epsilon}\right)\left(\mathrm{ch}\left(f^*E\right) - \mathrm{rk}\left(\mathbf{C}^{\mathrm{rk}(E)}\right)\right), \left[\widehat{M_\epsilon}\right]\right\rangle$$

$$= \deg(f)\left\langle \mathrm{ch}(E), [S^n(1)]\right\rangle,$$

a contradiction with $\deg(f) \neq 0$. Q.E.D.

Area enlargeability and positive scalar curvature

- **Schoen-Yau, Gromov-Lawson (1980)** There is no metric of positive scalar curvature on T^n. Moreover, there is no metric of positive scalar curvature on $T^n \# N$ for closed spin N.

- **Lohkamp (1998) :** There is no metric of positive scalar curvature on $T^n \# N$ for closed N implies the positive mass theorem for N.

- Schoen-Yau (1979) first proved positive mass theorem for any closed N with $\dim N \leq 7$ by using minimal hypersurface method. Witten (1981) first proved positive mass theorem for any closed spin N. Schoen-Yau (2017/2021) presented a proof of positive mass theorem for all closed N using minimal hypersurface methods.
- No Dirac operator proof for nonspin N, even for $T^4 \# \mathbf{C}P^2$.

The noncompact case

- Back to Gromov-Lawson's original result
- **Gromov-Lawson (1983).** If (M, g^{TM}) is a complete spin area enlargeable manifold, then the scalar curvature k^{TM} of g^{TM} can not have a positive lower bound, i.e.,

$$\inf\left(k^{TM}\right) \leq 0.$$

- With our new relative index theorem for deformed Dirac operators, which does not require the uniform postivity of the scalar curvature near infinity, one may ask whether one can improve

$$\inf\left(k^{TM}\right) \leq 0$$

to

$$\inf\left(k^{TM}\right) < 0?$$

- No general result available.

The noncompact case

- **X. Wang - Zhang (Chin. Ann. Math. 2022).** If M is a closed spin area enlargeable manifold and N is a noncompact spin manifold, then there is no complete metric of positive scalar curvature on $M\#N$.

- **Corollary.** For any noncompact spin N, there is no complete metric of positive scalar curvature on $T^n\#N$.

- When $n = 3$, the above Corollary is due to Lesourd-Unger-Yau (2020).
- When $3 \leq n \leq 7$, it was proved by Chodosh-Li (2020) without assuming that N is spin. It is closely related to the positive mass theorem in noncompact setting.
- General case still open.

Connes vanishing theorem on foliations

- **Connes (1986)** Let (M, F) be a closed oriented foliation such that the integrable subbundle F is spin. If there is a metric on TM such that its restriction to each leaf has positive scalar curvature, then $\widehat{A}(M) = 0$.

- When $F = TM$, Lichnerowicz vanishing theorem.
- Connes's proof makes use of cyclic cohomology and the Connes-Skandalis longitudinal index theorem.

- **Natural Question :** a possible direct geometric proof ?

- **Answer :** this can be done by using deformations of (sub) Dirac operators ...

Positive scalar curvature on foliations

- **Zhang (2017)** Let (M, F) be a closed spin foliated manifold. If there is a metric on TM such that its restriction to each leaf has positive scalar curvature, then $\widehat{A}(M) = 0$.
- **Corollary.** Let (T^n, F) be a foliation on T^n. Then there is no metric on T^n such that its restriction to each leaf has positive scalar curvature.
- When $F = T(T^n)$, Schoen-Yau and Gromov-Lawson.

Positive scalar curvature on foliations

- **G. Su - X. Wang - Zhang (Crelle 2022)** Let (M, g^{TM}) be a complete area enlargeable Riemannian manifold, $F \subseteq TM$ an integrable subbundle. If either TM or F is spin, then there is no $\delta > 0$, such that the restriction of g^{TM} to each leaf has scalar curvature $\geq \delta$.

- When $F = TM$, Gromov-Lawson (1983)

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- **Open question** Let (M, F) be a foliated manifold. If there is a metric on TM such that its restriction to each leaf has positive scalar curvature, then whether there is a metric on TM of positive scalar curvature ?

Thanks!