Nearly invariant subspaces for shift semigroups

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Contents

1. Invariant subspace problem

Let $\mathcal{B}(\mathcal{H})$ (or $\mathcal{B}(X)$) denote the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ (or a Banach space X).

The invariant subspace problem is the most important open problem in operator theory.

Recall that a subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is invariant with respect to an operator $T \in \mathcal{B}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$. The set $\mathcal{M}$ is nontrivial if

$$\{0\} \neq \mathcal{M} \neq \mathcal{H}.$$

Invariant subspace problem

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ (or a general Banach space X) of dimension ≥ 2. Does there exist a nontrivial closed subspace invariant with respect to T?

1. Invariant subspace problem

Remark

(a) If $\mathcal{H}$ is nonseparable and f is any nonzero vector of $\mathcal{H}$, then the vectors $f, Tf, T^2f, \ldots$ span a nontrivial closed subspace invariant with respect to T;

(b) If $dim\mathcal{H} < \infty$, then T has at least one eigenvalue and the corresponding eigenvector generates an invariant subspace of dimension 1.

The invariant subspace problem has sense only for separable infinite dimensional spaces.

1. Invariant subspace problem

—Some positive results

• von Neumann, 1930 - for compact operators on Hilbert spaces (unpublished).

• Aronszajn and Smith, 1954 - for compact operators on Banach spaces.

• Lomonosov, 1973- for operators commuting with a compact operator.

• Argyros and Haydon, 2011 - example of an infinite dimensional Banach space s.t. every bounded linear operator is a compact perturbation of a multiple of identity.

[1] Aronszajn N. Smith K T. Invariant subspaces of completely continuous operators, Ann. of Math. 60(2) (1954) 345-350.
[2] Lomonosov V I. Invariant subspaces of the family of operators that commute with a completely continuous operator, Funktsional. Anal. i Prilozhen, 7 (3) (1973) 55-56.
[3] Argyros S A. Haydon R G. A hereditarily indecomposable L^{∞}-space that solves the scalar-plus-compact problem, Acta Math 206(2011) 1-54.

1. Invariant subspace problem

—Several negative examples

- Enflo, 1987 - first example of a bounded operator on non-reflexive Banach space without nontrivial invariant subspaces.
- Read, 1991, 1997 - strictly singular operators and quasinilpotent operators on ℓ_1 without nontrivial invariant subspaces.
- Problem is still open for ℓ_2, separable Hilbert spaces, reflexive Banach spaces.

[4] Enflo P. On the invariant subspace problem for Banach spaces, Acta Math. 158 (34) (1987) 213-313.
[5] C.J. Read, Strictly singular operators and the invariant subspace problem, Studia Math. 132 (3)(1991) 203-226.
[6] Read C J. Quasinilpotent operators and the invariant subspace problem, J. Lond. Math. Soc. 56 (3)(1997) 595-606.

2. Related subspaces

- Reducing subspace $\mathcal{M}$ for operator $T \in \mathcal{B}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$ and $T\mathcal{M}^{\perp} \subset \mathcal{M}^{\perp}$. $\Leftrightarrow$ the orthogonal projection $P_{\mathcal{M}}$ onto $\mathcal{M}$ satisfies $P_{\mathcal{M}}T = TP_{\mathcal{M}}$.

- Quasi-invariant subspace $\mathcal{M}$ in Fock space $\mathcal{F}^2$ if $pf \in \mathcal{F}^2$ implies $pf \in \mathcal{M}$ for any $f \in \mathcal{M}$ and any polynomial $p \in \mathcal{F}^2$. (Introduced for Fock space $\mathcal{F}^2$), where

$$\mathcal{F}^2 = \{f \in H(\mathbb{C}) : \; \|f\|^2_{\mathcal{F}^2} := \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \frac{dA(z)}{\pi} < \infty\}.$$

[7] J. Hu, S. Sun, X. Xu and D. Yu, Reducing subspace of analytic Toeplitz operators on the Bergman space, Integr. Equ. Oper. Theory 49(3)(2004)387-395.
[8] K. Guo, D. Zheng, Invariant subspaces, quasi-invariant subspaces and Hankel operators, J. Funct. Anal. 187 (2001) 308-342.
[9] X. Chen, K. Guo, S. Hou, Analytic Hilbert spaces over the complex plane, J. Math. Anal. Appl. 268(2)(2002) 684-700.

2. Related subspaces

- Almost invariant subspace $\mathcal{M}$ if there exists a finite dimensional subspace $\mathcal{F}$ such that $T(\mathcal{M}) \subset \mathcal{M} + \mathcal{F}$.

Equivalent Question: Given a $T \in \mathcal{B}(X)$, can we perturb it by a finite rank operator F such that $T + F$ has an invariant subspace?

Theorem (Tcaciuc, Duke Mathematical Journal, 2019)

Let X be an infinite dimensional complex Banach space and $T \in \mathcal{B}(X)$. Then there exists a bounded operator F of rank at most one such that $T + F$ has an invariant half-space.

(half-space: it is infinite dimension and infinite codimension.)

[10] G. Androulakis, A.I. Popov, A. Tcaciuc, V.G. Troitsky, Almost invariant half-spaces of operators on Banach spaces, Integr. Equ. Oper. Theory 65 (4) (2009) 473-484.

[11] A. Tcaciuc, The invariant subspace problem for rank-one perturbations, Duke Mathematical Journal, (2019) Doi: 10.1215/ 00127094-2018-0071.

3. Hardy spaces

Hardy space $H^p(\mathbb{D})$

For $1 \le p < \infty$, $H^p(\mathbb{D})$ contains all $f \in H(\mathbb{D})$ such that

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

The space $H^\infty(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\infty = \sup_{|z|<1} |f(z)| < \infty.$$

Vector-valued Hardy space $H^p(\mathbb{D}, \mathbb{C}^l)$

$H^p(\mathbb{D}, \mathbb{C}^l)$ consists of all analytic mapping $F : \mathbb{D} \to \mathbb{C}^l$ such that

$$\|F\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|^p d\theta \right)^{1/p} < \infty.$$

3. Hardy spaces

Hardy Hilbert space $H^2(\mathbb{D})$

$H^2(\mathbb{D})$ is the Hardy space defined on $\mathbb{D}$ behaving

$$H^2(\mathbb{D}) = \{f \in H(\mathbb{D}), f(z) = \sum_{k=0}^{\infty} a_k z^k,\ \|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty\} \sim \ell^2.$$

$$\langle f, g\rangle = \sum_{k=0}^{\infty} a_k \overline{b_k} \text{ for } g(z) = \sum_{k=0}^{\infty} b_k z^k \in H^2(\mathbb{D}).$$

Recall the biholomorphism $A:\ \mathbb{D} \to \mathbb{C}_+$ is defined as $A(z) = \frac{1-z}{1+z}$.

Hardy space $H^2(\mathbb{C}_+)$

$H^2(\mathbb{C}_+)$ is the Hardy space defined on the right half-plane $\mathbb{C}_+ = \{s = x + iy,\ x > 0\}$ contains all analytic functions $f : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|f\|^2_{H^2(\mathbb{C}_+)} = \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 dy < \infty.$$

3. Hardy spaces

Inner-outer Factorization

Let f be a nonzero function in $H^1(\mathbb{D})$. Then f has a factoriza -tion $f = \theta \cdot u$, where θ is inner and u is outer. This factorization is unique up to a constant of modulus 1.

Inner function and outer function

An inner function is an H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$.

An outer function is a function $f \in H^1$ that has the form

$$f(z) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{iw}+z}{e^{iw}-z} k(e^{iw})dw\right),\ z \in \mathbb{D},$$

where k is a real-valued integrable function and $|\alpha| = 1$.

3. Hardy spaces

Examples of outer functions

All invertible functions in $H^\infty(\mathbb{D})$. Polynomials whose zeroes all lie outside $\overline{\mathbb{D}}$. Further all polynomials whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$ are outer functions, such as $1 - z$, $1 + z$ and so on.

Examples of inner functions

All automorphisms $(z - a)/(1 - \bar{a}z)$ with a, $z \in \mathbb{D}$. More generally, the finite and infinite degree Blaschke products

$$B_n(z) = \alpha \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z},$$

where $|\alpha| = 1$ and $|z_j| < 1$ for $j = 1, \cdots, n$;

$$B(z) = z^m \prod_{z_k \neq 0} \frac{\overline{z_k}}{|z_k|} \frac{z - z_k}{1 - \overline{z_k} z},$$

where $\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$.

4. Nearly invariant spaces

4.1 Unilateral shift and invariant subspaces

Unilateral shift

Let $S: H^2(\mathbb{D}) \to H^2(\mathbb{D})$ denote the unilateral shift

$$[Sf](z) = zf(z).$$

Its adjoint operator on $H^2(\mathbb{D})$ is $[S^*f](z) = (f(z) - f(0))/z$.

Beurling's Theorem, 1949

Let $\mathcal{K}$ be a non-zero subspace of $H^2(\mathbb{D})$ that is invariant under S. Then $\mathcal{K} = \theta H^2(\mathbb{D})$ for some inner function θ, which is unique to within a constant of modulus 1.

Remark

The proper S^*-invariant subspace is model space

$$K_\theta := (\theta H^2)^\perp = H^2 \ominus \theta H^2.$$

4. Nearly invariant spaces

Toeplitz operator on $H^2(\mathbb{D})$

For $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is

$$(T_\phi f)(\lambda) = P_{H^2}(\phi \cdot f)(\lambda) = \int_{\mathbb{T}} \frac{\phi(\zeta) f(\zeta)}{1 - \overline{\zeta}\lambda} dm(\zeta),$$

where P_{H^2} is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$.

$$L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus \overline{H_0^2(\mathbb{D})} = H^2(\mathbb{D}) \oplus \overline{zH^2(\mathbb{D})},$$

where $H^2(\mathbb{D}) = \bigvee\{z^n : n \geq 0\}$ and $\overline{H_0^2(\mathbb{D})} = \bigvee\{z^n : n < 0\}$.

Remark

Let $\phi(z) = z$, it follows that $T_z = S$ and $T_z^* = S^*$.

[11] S. Garcia, J. Mashreghi, W.T. Ross, Introduction to Model Spaces and Their Operators, Cambridge: Cambridge University Press (2016).

4. Nearly invariant spaces

4.2 Nearly S^*-invariant subspace in $H^2(\mathbb{D})$

Definition (Hitt, Pac. J. Math. 1988)

A closed subspace $\mathcal{M} \subset H^2(\mathbb{D})$ is nearly S^*-invariant if whenever $f \in \mathcal{M}$ and $f(0) = 0$, then $S^* f \in \mathcal{M}$. [weakly invariant]

Remark

$\mathcal{M}$ is S^* invariant if and only if $\mathcal{M}$ is nearly S^* invariant and $1 \in \mathcal{M} + z\mathcal{M}$.

[12] D. Hitt, Invariant subspaces of H^2 of an annulus, Pac. J. Math. 134 (1)(1988) 101-120.

[13] D. Sarason, Nearly invariant subspaces of the backward shift, In: Contributions to Operator Theory and Its Applications (Mesa, AZ, 1987), 481-493. Oper. Theory Adv. Appl., vol. 35. Birkhäuser, Basel (1988).

4. Nearly invariant spaces

Theorem (Hitt, 1988; Sarason, 1988)

The nearly S^*-invariant subspaces have the form $\mathcal{M} = uK$, with $u \in \mathcal{M}$ of the unit norm, $u(0) > 0$ and u orthogonal to all eleme -nts of $\mathcal{M}$ vanishing at the origin, K an S^*-invariant subspace, and the operator multiplication by u is isometric from K into $H^2(\mathbb{D})$.

Example

There is a nearly S^*-invariant subspace in $H^2(\mathbb{D})$

$$\mathcal{M} = \varphi_{\frac{1}{2}}(z)K_{z^n} = \frac{\frac{1}{2} - z}{1 - \frac{1}{2}z} \cdot \mathrm{span}\{1, z, z^2, \cdots, z^{n-1}\}.$$

4. Nearly invariant spaces

4.3 Nearly T_θ^*-invariant subspace in $H^2(\mathbb{D})$

Definition

A closed subspace $\mathcal{M} \subset H^2(\mathbb{D})$ is nearly T_θ^*-invariant if

$$f \in \mathcal{M},\ T_\theta^* f = \overline{\theta} f \in H^2(\mathbb{D}) \ \Rightarrow\ T_\theta^* f \in \mathcal{M}.$$

Example

Let $kerT_\phi$ denote the Toeplitz kernel of $T_\phi :\ H^2(\mathbb{D}) \to H^2(\mathbb{D})$. Then $kerT_\phi$ is nearly T_θ^* invariant for any inner function θ.

proof. Suppose $f \in kerT_\phi$ such that $\overline{\theta} f \in H^2$. It follows that

$$P_{H^2}(\phi f) = 0 \ \Rightarrow\ \phi f \in \overline{H_0^2} = \overline{zH^2}.$$

That means $\phi f = \overline{z}\overline{h}$ for some $h \in H^2$. Hence

$$\phi(\overline{\theta} f) = \overline{\theta}\overline{z}\overline{h} \in \overline{H_0^2} \ \Rightarrow\ \overline{\theta} f \in kerT_\phi.$$

4. Nearly invariant spaces

4.4 Nearly T^{-1}-invariant subspace in $\mathcal{H}$

Definition (Liang, Partington, CAOT, 2021)

Let $T \in \mathcal{B}(\mathcal{H})$ be left invertible. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is nearly T^{-1}-invariant if for every $f \in \mathcal{H}$ such that $Tf \in \mathcal{M}$, it holds that $f \in \mathcal{M}$.

Remark

Let $T = T_\theta : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ with an inner function θ, then the nearly T^{-1} invariance is nearly T_θ^* invariance.

[14] Liang Y, Partington J R. Nearly invariant subspaces for operators in Hilbert spaces. Complex Anal Oper Theory, 2021, 15: 5, 17pp

4. Nearly invariant spaces

Theorem (Liang, Partington, CAOT, 2021)

Let $\mathcal{M} \subset H^2(\mathbb{D})$ is a nonzero nearly $T^*_{B_m}$ invariant subspace with a degree-m Blaschke product B_m. Let the matrix

$$G_0(z) := [g_1(z), g_2(z), \cdots, g_l(z)]^t,$$

contain an orthonormal basis $(g_i(z))_{i=1}^l$ of $\mathcal{M} \ominus (\mathcal{M} \cap B_m H^2(\mathbb{D}))$. Then there exist a nonnegative integer $l' \le l$ and an operator-valued inner function $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{l'}, \mathbb{C}^l))$, unique up to unitary equivalence, s.t.

$$\mathcal{M} = \{f : \exists h \in H^2(\mathbb{D}, \mathbb{C}^l) \ominus \Phi H^2(\mathbb{D}, \mathbb{C}^{l'}), f(z) = h(T_{B_m})G_0(z)\}.$$

[14] Liang Y, Partington J R. Nearly invariant subspaces for operators in Hilbert spaces. Complex Anal Oper Theory, 2021, 15: 5, 17pp

4. Nearly invariant spaces

Question How to characterize the nearly T_B^* invariant subspace in $H^2(\mathbb{D})$ for an infinite degree Blaschke product B?

Caradus, Proc. Amer. Math. Soc. 1969

For an infinite degree Blaschke product B, the Toeplitz operator $T_B^* : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is universal and it is similar to the backward shift $S(1)^*$ on $L^2(0,\infty)$, given by $S(1)^* f(t) = f(t+1)$.

The shift semigroup

In general, the shift semigroup $S(t): \ L^2(0,\infty) \to L^2(0,\infty)$ with $t \geq 0$ is defined by

$$(S(t)f)(\zeta) = \begin{cases} 0, & \zeta \leq t, \\ f(\zeta - t), & \zeta > t. \end{cases} \tag{1}$$

And the adjoint semigroup $\{S(t)^*\}_{t\geq 0}$ is $(S(t)^* f)(\zeta) = f(\zeta + t)$.

[15] S.R. Caradus, Universal operators and invariant subspaces, Proc. Amer. Math. Soc. 23 (1969), 526-527.

4. Nearly invariant spaces

Commutative Diagrams

$$\begin{array}{ccc}
L^2(0,\infty) & \xrightarrow{S(t)} & L^2(0,\infty) \\
\Big\downarrow \mathcal{L} & & \Big\downarrow \mathcal{L} \\
H^2(\mathbb{C}_+) & \xrightarrow{M(t)} & H^2(\mathbb{C}_+) \\
\Big\downarrow V^{-1} & & \Big\downarrow V^{-1} \\
H^2(\mathbb{D}) & \xrightarrow{T(t)} & H^2(\mathbb{D}).
\end{array}$$

Here

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)dt, \;\; (V^{-1}g)(z) = \frac{2\sqrt{\pi}}{1+z} g\left(A(z)\right),$$

$$(M(t)g)(s) = e^{-st} g(s), \;\; (T(t)h)(z) = \phi^t(z)h(z),$$

with $\phi^t(z) := \exp\left(-t\frac{1-z}{1+z}\right)$.

5. Nearly invariant subspaces for C_0-semigroup

C_0-semigroup

A family $\{T(t)\}_{t\geq 0}$ in $\mathcal{B}(\mathcal{H})$ is called a C_0-semigroup if $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$ and $\lim\limits_{t\to 0} T(t)x = x$ for any $x \in \mathcal{H}$.

Nearly $\{T(t)^*\}_{t\geq 0}$ invariant subspace

Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup in $\mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{H}$ be a subspace. If for every $x \in \mathcal{H}$ whenever $T(t)x \in \mathcal{N}$ for some $t > 0$, then $x \in \mathcal{N}$, we call $\mathcal{N}$ a nearly $\{T(t)^*\}_{t\geq 0}$ invariant subspace.

Remark

Every Toeplitz kernel in $H^2(\mathbb{C}_+)$ or $H^2(\mathbb{D})$ is nearly $\{M(t)^*\}_{t\geq 0}$ or $\{T(t)^*\}_{t\geq 0}$ invariant in $H^2(\mathbb{C}_+)$ or $H^2(\mathbb{D})$, respectively.

[16] Liang, Y., Partington, J.R. Nearly invariant subspaces for shift semigroups. Sci. China Math. (2022). https://doi.org/10.1007/ s11425-020-1915-y

5. Nearly invariant subspaces for C_0-semigroup

The smallest (cyclic) nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace

Let $\mathcal{N} \subseteq L^2(0,\infty)$ be a nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace, and denote the smallest nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace in $\mathcal{N}$ containing some nonzero vector f by $[f]_s$. There are two cases.

(i) There is no function $f \in \mathcal{N}$, apart from the zero function, for which there exists some $\delta > 0$ with $f = 0$ almost everywhere on $(0,\delta)$. In this case, $\mathcal{N}$ is a trivial nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace and $[f]_s = \mathbb{C}f$ for all $f \in \mathcal{N}$.

(ii) There are a $\delta > 0$ and a function $f \in \mathcal{N}$ that vanishes almost everywhere on $(0,\delta)$ and not on $(0,\delta+\epsilon)$ for any $\epsilon > 0$. Since $S(\delta)S(\delta)^*f = f \in \mathcal{N}$, the near $\{S(t)^*\}_{t\geq 0}$ invariance implies $g := S(\delta)^*f \in \mathcal{N}$. Meanwhile, we have $S(\lambda)g = S(\delta-\lambda)^*f \in \mathcal{N}$ for all $0 \leq \lambda \leq \delta$. So

$$[f]_s = \bigvee\{S(\delta-\lambda)^*f,\ 0 \leq \lambda \leq \delta\}.$$

5. Nearly invariant subspaces for C_0-semigroup

Main Example 1. $f(\zeta) = e_\delta(\zeta) := e^{-\zeta}\chi_{(\delta,\infty)}(\zeta)$ with $\delta > 0$.

The orthonormal basis for $L^2(0,\infty)$

It is known that $\{\sqrt{2\pi}p_n(t)e^{-t}\}_{n=0}^{\infty}$ forms an orthonormal basis for $L^2(0,\infty)$, with $p_n(t) = \pm L_n(2t)/\sqrt{\pi}$ (a real polynomial of degree n) and L_n denotes the Laguerre polynomial

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}).$$

Proposition 5.1

In $L^2(0,\infty)$, the smallest nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace containing e_δ with some $\delta > 0$ has the form

$$[e_\delta]_s := \bigvee\{e_\lambda,\ 0 \leq \lambda \leq \delta\} = L^2(0,\delta) + \mathbb{C}e^{-\zeta}.$$

5. Nearly invariant subspaces for C_0-semigroup

Proposition 5.2

In $H^2(\mathbb{C}_+)$, the Laplace transform of $[e_\delta]_s$ is

$$\mathcal{L}([e_\delta]_s) = \bigvee\{\frac{e^{-\lambda s}}{1+s},\ 0 \le \lambda \le \delta\} = K_{e^{-\delta s}} + \mathbb{C}\frac{1}{1+s} = K_{\frac{1-s}{1+s}e^{-\delta s}},$$

where $K_{e^{-\delta s}}$ and $K_{\frac{1-s}{1+s}e^{-\delta s}}$ are model spaces in $H^2(\mathbb{C}_+)$.

Proposition 5.3

In $H^2(\mathbb{D})$, it holds that

$$V^{-1}(\mathcal{L}([e_\delta]_s)) = \bigvee\{\phi^\lambda,\ 0 \le \lambda \le \delta\} = K_{z\phi^\delta},$$

where $\phi^\lambda(z) := \exp\left(-\lambda\frac{1-z}{1+z}\right)$.

5. Nearly invariant subspaces for C_0-semigroup

Corollary 5.4

In $H^2(\mathbb{D})$, it holds that

$$\bigvee\{\phi^\lambda,\ 0 \le \lambda < \infty\} = H^2(\mathbb{D}).$$

Corollary 5.5

In $H^2(\mathbb{C}_+)$, it holds that

$$\bigvee\{\frac{e^{-\lambda s}}{1+s},\ 0 \le \lambda < \infty\} = H^2(\mathbb{C}_+).$$

5. Nearly invariant subspaces for C_0-semigroup

Main Example 2. $f(\zeta) = f_{\delta,1}(\zeta) := (\zeta - \delta)e_\delta(\zeta)$ with $\delta > 0$.

Using the Laplace transform, it holds that

$$e^\delta \mathcal{L}(f_{\delta,1})(s) = \frac{e^{-\delta s}}{(1+s)^2}.$$

The smallest nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace containing the vector $f_{\delta,1}$ in $\mathcal{N}$ is

$$[f_{\delta,1}]_s = \bigvee \{(\zeta - \lambda)e_\lambda,\ 0 \leq \lambda \leq \delta\}.$$

Proposition 5.6

In $H^2(\mathbb{D})$, it holds that

$$V^{-1}(\mathcal{L}([f_{\delta,1}]_s)) = \bigvee \{(1+z)\phi^\lambda,\ 0 \leq \lambda \leq \delta\} = K_{z^2\phi^\delta}.$$

5. Nearly invariant subspaces for C_0-semigroup

Theorem 5.7

The Laplace transform of the smallest nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspace in $H^2(\mathbb{C}_+)$ containing $f_{\delta,1}$ with some $\delta > 0$ has the form

$$\mathcal{L}([f_{\delta,1}]_s) = \bigvee\{\frac{e^{-\lambda s}}{(1+s)^2},\ 0 \leq \lambda \leq \delta\} = K_{\left(\frac{1-s}{1+s}\right)^2 e^{-\delta s}},$$

where $K_{\left(\frac{1-s}{1+s}\right)^2 e^{-\delta s}}$ is a model space in $H^2(\mathbb{C}_+)$.

5. Nearly invariant subspaces for C_0-semigroup

Main Example 3. $f(\zeta) = f_{\delta,n}(\zeta) := \frac{(\zeta-\delta)^n}{n!} e_\delta(\zeta)$ with $\delta > 0,\ n \geq 0$.

By the Laplace transform we have

$$e^\delta \mathcal{L}(f_{\delta,n})(s) = \frac{e^{-\delta s}}{(1+s)^{n+1}}.$$

Theorem 5.8

For any nonnegative integer n and $\delta > 0$, the followings hold.

(1) In $H^2(\mathbb{D})$, it holds that the smallest nearly $\{T(t)^*\}_{t\geq 0}$ invariant subspace $\bigvee\{(1+z)^n\phi^\lambda,\ 0 \leq \lambda \leq \delta\} = K_{z^{n+1}\phi^\delta}$;

(2) In $H^2(\mathbb{C}_+)$, it holds that the smallest nearly $\{M(t)^*\}_{t\geq 0}$ invariant subspace $\bigvee\{\frac{e^{-\lambda s}}{(1+s)^{n+1}},\ 0 \leq \lambda \leq \delta\} = K_{\left(\frac{1-s}{1+s}\right)^{n+1} e^{-\delta s}}$.

6. Further results and questions

Further results

Define
$$c(g) := \overline{gK_{z\phi^\delta}} = \bigvee\{g\phi^\lambda,\ 0 \le \lambda \le \delta\}$$
for a more general function $g \in L^\infty(\mathbb{T})$.

Theorem 6.1: The nearly S^* invariant property of $\overline{gK_\theta}$

Let $g \in H^\infty(\mathbb{D})$ with $g(0) \neq 0$ and θ a non-constant inner function. Then $\overline{gK_\theta}$ is nearly S^* invariant, and so by Hitt's theorem it can be written as hK, where K is S^*-invariant (either a model space or $H^2(\mathbb{D})$ itself) and $h \in H^2(\mathbb{D})$ is a function such that multiplication by h is isometric on K.

6. Further results and questions

Proposition 6.2

Let $\widetilde{p}_N(z) := \prod_{j=1}^{N}(z + w_j)$ with $w_j \in \mathbb{T}$, $j = 1, \cdots, N$, it follows that

$$c(\widetilde{p}_N) + \phi^\delta K_{z^N} = K_{z^{N+1}\phi^\delta}.$$

Hence $c(\widetilde{p}_N)$ has codimension at most N in $K_{z^{N+1}\phi^\delta}$.

Theorem 6.3

Suppose $g(z) = \widetilde{p}_N(z)h(z)$ with $\widetilde{p}_N(z) := \prod_{j=1}^{N}(z + w_j)$, where $w_j \in \mathbb{T}$, $j = 1, \cdots, N$, and h is an invertible rational function in $L^\infty(\mathbb{T})$. Then $c(g)$ has codimension at most N in $hK_{z^{N+1}\phi^\delta}$, that is,

$$c(g) + h\phi^\delta K_{z^N} = hK_{z^{N+1}\phi^\delta}.$$

6. Further results and questions

Theorem 6.4

Let $g \in H^2(\mathbb{C}_+)$ be rational with m zeros on the imaginary axis and let $n > m$ such that $s^{n-m}g(s)$ tends to a finite nonzero limit at ∞. Then g can be written as $g = G_1 G_2$, where G_1 is rational and invertible in $L^\infty(i\mathbb{R})$ and $G_2(s) = \prod_{k=1}^{m}(s - y_k)/(s+1)^n$ with all $y_k \in i\mathbb{R}$. Then it holds that

$$\bigvee\{ge^{-\lambda s},\ 0 \le \lambda \le \delta\} + G_1 e^{-\delta s} K_{(\frac{1-s}{1+s})^{n-1}} = G_1 K_{(\frac{1-s}{1+s})^n e^{-\delta s}}.$$

6. Further results and questions

Further questions

- 1. Are there more general examples?
- 2. What is the complete characterization for nearly $\{S(t)^*\}_{t\geq 0}$ invariant subspaces in $L^2(0,\infty)$?
- 3. How to give the nearly T_B^* invariant subspaces for an infinite Blaschke product B?

Thanks!