Some configurations of conics in $PG(2, q)$ with q even

Seon Jeong Kim

Gyeongsang National University
Korea

Shanghai, July 2-6, 2018

A quadratic polynomial over a field F means

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz,$$

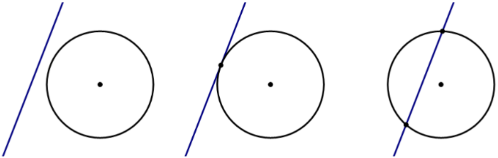
where $a_i \in F$, $i = 1, \ldots, 6$.

A conic means the zero set of a quadratic polynomial which is absolutely irreducible (i.e., irreducible over the algebraic closure of the ground field F).

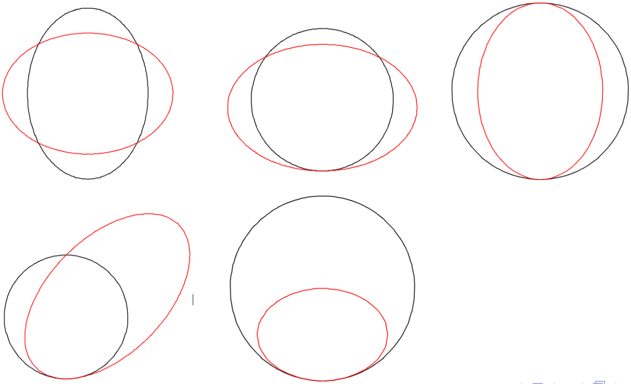
On the projcective plane over F, any two conics are projectively equivalent to each other, i.e., there is a projective transformation between them.

Relative position of a line and a conic

Relative position of a line and a circle



Relative position of two conics (over the algebraic closure)



Notations

$\mathbb{F}_q$, the finite field with q elements

$\mathbb{P}^2$, the projective plane over the algebraic closure of $\mathbb{F}_q$

For an algebraic set (the zero set of polynomials) X in $\mathbb{P}^2$,

$X(\mathbb{F}_q) := \{(\alpha, \beta, \gamma) \in X \mid \alpha, \beta, \gamma \in \mathbb{F}_q\}$, the set of $\mathbb{F}_q$-points of X

$N_q(X) := {}^{\#}X(\mathbb{F}_q)$, the number of $\mathbb{F}_q$-points of X

For $u, v, w \in \mathbb{F}_q$, not all zero,
$[u, v, w] := \{(x, y, z) \in \mathbb{P}^2 \mid ux + vy + wz = 0\}$, an $\mathbb{F}_q$-line.

$\theta_2 := {}^{\#}\mathbb{P}^2(\mathbb{F}_q) = q^2 + q + 1$, the number of $\mathbb{F}_q$-points on the plane which is equal to that of $\mathbb{F}_q$-lines.

Preliminaries

1. Every $\mathbb{F}_q$-line contains $q+1$ $\mathbb{F}_q$-points.
2. There are exactly $q+1$ $\mathbb{F}_q$-lines through any $\mathbb{F}_q$-points, whose union contains $\mathbb{P}^2(\mathbb{F}_q)$.
3. Every conic contains $q+1$ $\mathbb{F}_q$-points.
4. The number of $\mathbb{F}_q$-points in the intersection of a line and a conic is 0, 1, or 2. (Definition. A line ℓ is called a *i-line* of a set S if $|\ell \cap S(\mathbb{F}_q)| = i$.)
5. For a conic C, $|\mathcal{L}_0(C)| = \frac{q(q-1)}{2}$, $|\mathcal{L}_1(C)| = q+1$, and $|\mathcal{L}_2(C)| = \frac{q(q+1)}{2}$, where $\mathcal{L}_i(C)$ means the set of i-lines of C. (A 1-line means a tangent line.)
6. For $q \geq 4$, every 5 points, any three of them are not collinear, determine a unique conic.
7. For even q, all $q+1$ tangent lines to a conic C pass through the common point N, which is called the nucleus of the conic C.

Conics with the common nucleus

From now on, we only consider for even $q = 2^m$.

We denote by $C(a_1, a_2, a_3, a_4, a_5, a_6)$ the conic defined by the quadratic equation

$$a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz = 0.$$

1. The nucleus of the conic $C(a_1, a_2, a_3, a_4, a_5, a_6)$ is (a_6, a_5, a_4).
2. We may let the equation of a conic with nucleus $(0, 0, 1)$ is $a_1x^2 + a_2y^2 + a_3z^2 + xy = 0$. (Here $a_3 \neq 0$, since the polynomial is absolutely irreducible.)
3. The number of all conics with the common nucleus is $q^2(q-1)$.
4. The number of all conics on the projective plane over $\mathbb{F}_q$ is $q^5 - q^2$.

Spectrum and Standard Equations

R. H. F. Denniston, R. Mathon and many researchers used such conics to construct maximal arcs on the plane.

For a set S, we let $s_i := |\mathcal{L}_i(S)|$, the number of i-lines of S. The sequence $\{s_i\}$ is called the spectrum of S.
A set S is called an (n, r)-arc if $|S| = n$ and $r = \max\{i \mid s_i \neq 0\}$.
A set S is called a maximal (n, r)-arc if $s_i = 0$ for any $0 < i < r$.

Standard equations for a set S

(1) $\sum_{i=0}^{q} s_i = q^2 + q + 1$ (the number of all lines on the plane).

(2) $\sum_{i=0}^{q} i \cdot s_i = (q^2 - q + 1) \cdot (q + 1)$ (the sum of $|\ell \cap S|$ for all lines on the plane).

(3) $\sum_{i=2}^{q} \binom{i}{2} s_i = \binom{q^2-q+1}{2}$ (counting the number of elements in the set $\{(\{P, Q\}, \overline{PQ}) \mid P, Q \in S \text{ and } P \neq Q\}$ in two ways).

Zeros of a quadratic equation

We use the trace function $Tr : \mathbb{F}_q \to \mathbb{F}_2 = \{0, 1\}$ defined by

$$Tr(x) := x + x^2 + x^4 + \cdots + x^{\frac{q}{2}}.$$

Lemma.

(1) The quadratic equation $t^2 + t + \delta = 0$ over $\mathbb{F}_q$ has a solution in $\mathbb{F}_q$ if and only if $Tr(\delta) = 0$. In this case, it has 2 distinct solutions t_0 and $1 + t_0$ where $t_0 = k\delta^2 + (k + k^2)\delta^4 + \cdots + (k + k^2 + \cdots + k^{\frac{q}{4}})\delta^{\frac{q}{2}}$ for an element $k \in \mathbb{F}_q$ with $Tr(k) = 1$.

(2) In general, an equation $at^2 + bt + c = 0$ with $a(\neq 0), b(\neq 0), c \in \mathbb{F}_q$ has a solution (actually 2 distinct zeros) in $\mathbb{F}_q$ if and only if $Tr(\frac{ac}{b^2}) = 0$.

(3) An equation $at^2 + bt + c = 0$ with $a(\neq 0), b, c \in \mathbb{F}_q$ has a double zero in $\mathbb{F}_q$ if and only if $b = 0$.

Intersection of two conics with the common nucleus

We consider a family of $q^2(q-1)$ conics with the common nucleus $N(0,0,1)$

$$ax^2 + by^2 + \lambda z^2 + xy = 0 \text{ with } \ a, b, \lambda \in \mathbb{F}_q, \ \lambda \neq 0,$$

which we denote by $F_{ab\lambda}$ simply.

Lemma. Let $F_{ab\lambda}$ and $F_{a'b'\lambda'}$ be two distinct conics.

(1) If $\lambda = \lambda'$, then they have one common $\mathbb{F}_q$-point of intersection multiplicity 4.

(2) Let $\lambda \neq \lambda'$. If $Tr(\frac{(a\lambda'+a'\lambda)(b\lambda'+b'\lambda)}{(\lambda+\lambda')^2}) = 0$, then they have two common $\mathbb{F}_q$-points of intersection multiplicity 2, respectrively. If $Tr(\frac{(a\lambda'+a'\lambda)(b\lambda'+b'\lambda)}{(\lambda+\lambda')^2}) = 1$, then they have no common $\mathbb{F}_q$-points.

Sum of two conics

For two conics $F_{ab\lambda}$ and $F_{a'b'\lambda'}$ with $\lambda \neq \lambda'$, we define another conic as

$$F_{ab\lambda} \oplus F_{a'b'\lambda'} := F_{a\oplus a', b\oplus b', \lambda\oplus\lambda'},$$

where $a \oplus a' = \frac{a\lambda + a'\lambda'}{\lambda+\lambda'}$, $b \oplus b' = \frac{b\lambda + b'\lambda'}{\lambda+\lambda'}$, $\lambda \oplus \lambda' = \lambda + \lambda'$.

Lemma.

(1) If $F_{ab\lambda}$ and $F_{a'b'\lambda'}$ with $\lambda \neq \lambda'$ have no common $\mathbb{F}_q$-points, then $F_{ab\lambda} \oplus F_{a'b'\lambda'}$ has no common $\mathbb{F}_q$-points with both of them.

(2) If $F_{ab\lambda}$ and $F_{a'b'\lambda'}$ with $\lambda \neq \lambda'$ have two common $\mathbb{F}_q$-points, then $F_{ab\lambda} \oplus F_{a'b'\lambda'}$ contains those two common $\mathbb{F}_q$-points.

Known Results of Denniston and Mathon

Theorem (Denniston, 1969) Let $\phi(x, y) = ax^2 + hxy + by^2 \in \mathbb{F}_q[x, y]$ be irreducible over $\mathbb{F}_q$. Let H be an additive subgroup of $\mathbb{F}_q$ of order r. Then the set $\{(x, y, z) \mid \phi(x, y) \in H\}$ is an $((r-1)(q+1)+1, r)$-arc, which is maximal.

Theorem (Mathon, 2002) Let $Tr(ab) = 1$. Let H be an additive subgroup of $\mathbb{F}_q$ of order r. Then the set $S_H = \cup_{\lambda \in H} F_{ab\lambda}$ is an $((r-1)(q+1)+1, r)$-arc, which is maximal.

Theorem (Mathon, 2002) Let $p(\lambda) = \sum_{i=0}^{d-1} a_i \lambda^{2^i-1}$ and $q(\lambda) = \sum_{i=0}^{d-1} b_i \lambda^{2^i-1}$ be polynomials with coefficients in $\mathbb{F}_{2^m}$. For an additive subgroup A of order 2^d in $\mathbb{F}_{2^m}$ let $\mathcal{F} = \{F_{p(\lambda)q(\lambda)\lambda} \mid \lambda \in A\}$ be the set of conics with a common nucleus F_0. If $Tr(p(\lambda)q(\lambda)) = 1$ for every $\lambda \in A$ then the set of points on all conics in $\mathcal{F}$ together with F_0 form a maximal $2^{m+d} - 2^m + 2^d, 2^d)$-arc $\mathcal{K}$ in $\mathbb{P}^2(F_{2^m})$. If both $p(\lambda)$, $q(\lambda)$ are of degree ≤ 1 in λ then $\mathcal{K}$ is a Denniston maximal arc.

Some Notations

We consider more family of conics or lines related to such conics.

To determine disjointness of two conics $F_{ab\lambda}, F_{a'b'\lambda'}$ with $\lambda \neq \lambda'$ or that of a conic $F_{ab\lambda}$ and a line $[u, v, w]$ with $w \neq 0$, we define the notations.

$$DCC(F_{ab\lambda}, F_{a'b'\lambda'}) := \frac{(a\lambda' + a'\lambda)(b\lambda' + b'\lambda)}{(\lambda + \lambda')^2}.$$

$$DLC([u, v, w], F_{ab\lambda}) := \frac{(aw^2 + \lambda u^2)(bw^2 + \lambda v^2)}{w^4}.$$

Note that a line $[u, v, 0]$ pass through the nucleus $N(0, 0, 1)$, so it is tangent to every conic $F_{ab\lambda}$. Also note that if $\lambda = \lambda'$, then $F_{ab\lambda}$ and $F_{a'b'\lambda'}$ meet at a $\mathbb{F}_q$-point of intersection multiplicity 4.

Union of disjoint conics with the common nucleus

Theorem. Let $\lambda, \lambda_1, \lambda_2 \in \mathbb{F}_q \setminus \{0\}$ and $\lambda_1 \neq \lambda_2$.

(1) $DCC(F_{a_1 b_1 \lambda_1}, F_{a_2 b_2 \lambda_2}) = DCC(F_{a_1 b_1 (\lambda_1 \lambda)}, F_{a_2 b_2 (\lambda_2 \lambda)})$.

(2) $DCC(F_{a_1 b_1 \lambda_1}, F_{a_2 b_2 \lambda_2}) + DCC(F_{a_1 b_1 \lambda_2}, F_{a_2 b_2 \lambda_1}) = a_1 b_1 + a_2 b_2$.

(3) $DCC(F_{a_1 b_1 \lambda_1}, F_{a_2 b_2 \lambda_2}) = DCC(F_{b_1 a_1 \lambda_1}, F_{b_2 a_2 \lambda_2})$.

(4) $DLC([u, v, w], F_{ab\lambda}) = DLC([v, u, w], F_{ba\lambda})$.

Corollary.

(1) Let $Tr(a_1 b_1) = Tr(a_2 b_2)$. Then $F_{a_1 b_1 \lambda_1} \cap F_{a_2 b_2 \lambda_2} = \emptyset$ if and only if $F_{a_1 b_1 \lambda_2} \cap F_{a_2 b_2 \lambda_1} = \emptyset$

(2) Let $Tr(a_1 b_1) \neq Tr(a_2 b_2)$. Then $F_{a_1 b_1 \lambda_1} \cap F_{a_2 b_2 \lambda_2} = \emptyset$ if and only if $F_{a_1 b_1 \lambda_2} \cap F_{a_2 b_2 \lambda_1} \neq \emptyset$

Automorphisms fixing $N(1, 0, 0)$

Let $M \in PGL(3, q)$ and ϕ_M be an automorphism determined by the matrix M. We consider the automorphisms fixing the nucleus $N(0, 0, 1)$ hence it preserves the family of conics with nucleus N.

$\{M \in PGL(3, q) \mid \phi_M(0, 0, 1) = (0, 0, 1)\} =$

$\left\{ \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix} \mid c_{11}c_{22} + c_{12}c_{21} \neq 0 \right\}$. For

$M = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix}$, let $\phi = \phi_M$ and

$d = \det(M) = c_{11}c_{22} + c_{12}c_{21} \neq 0$. Then

$\phi(x, y, z) = (c_{11}x + c_{12}y, c_{21}x + c_{22}y, c_{31}x + c_{32}y + z)$.

$\phi([u, v, w]) = [u', v', w']$, where

$$\begin{cases} u' = d^{-1}(c_{22}u + c_{21}v + (c_{21}c_{32} + c_{22}c_{31})w) \\ v' = d^{-1}(c_{12}u + c_{11}v + (c_{11}c_{32} + c_{12}c_{31})w) \\ w' = w \end{cases} .$$

Automorphisms fixing $N(1,0,0)$

$\phi(F_{ab\lambda}) = F_{a'b'\lambda'}$ where

$$\begin{cases} a' = d^{-1}(ac_{22}^2 + bc_{21}^2 + \lambda(c_{21}c_{32} + c_{22}c_{31})^2 + c_{22}c_{21}) \\ b' = d^{-1}(ac_{12}^2 + bc_{11}^2 + \lambda(c_{11}c_{32} + c_{12}c_{31})^2 + c_{12}c_{11}) \\ \lambda' = d\lambda \end{cases}.$$

$\phi^{-1} = \phi_{M^{-1}}$ where

$$M^{-1} = d^{-1}\begin{bmatrix} c_{22} & c_{12} & 0 \\ c_{21} & c_{11} & 0 \\ c_{21}c_{32} + c_{22}c_{31} & c_{11}c_{32} + c_{12}c_{31} & d \end{bmatrix}$$

$$\begin{aligned} &\phi(\{(x,y,z) \mid f(x,y,z) = 0\}) = \{\phi(x,y,z) \mid f(x,y,z) = 0\} \\ &= \{(x',y',z') \mid f(\phi^{-1}(x',y',z')) = 0\} \\ &= \{(x',y',z') \mid f(\frac{c_{22}x' + c_{12}y'}{d}, \frac{c_{21}x' + c_{11}y'}{d}, \\ &\qquad \frac{(c_{21}c_{32} + c_{22}c_{31})x' + (c_{11}c_{32} + c_{12}c_{31})y'}{d} + z') = 0\} \end{aligned}$$

Examples of automorphism groups

$$Aut(F_{001}) = \{M \in PGL(3,q) \mid \phi_M(F_{001}) = F_{001}\} =$$
$$\left\{ \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ \sqrt{c_{11}c_{21}} & \sqrt{c_{12}c_{22}} & 1 \end{bmatrix} \mid c_{11}c_{22} + c_{12}c_{21} = 1 \right\}.$$

Note that $\sqrt{a} = a^{\frac{q}{2}}$.

Let $\epsilon \in \mathbb{F}_q$ such that $Tr(\epsilon) = 1$.
$$Aut(F_{1\epsilon 1}) = \{M \in PGL(3,q) \mid \phi_M(F_{1\epsilon 1}) = F_{1\epsilon 1}\} =$$
$$\left\{ \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ \sqrt{1 + c_{11}c_{21} + c_{11}^2 + \epsilon c_{21}^2} & \sqrt{\epsilon + c_{12}c_{22} + c_{12}^2 + \epsilon c_{22}^2} & 1 \end{bmatrix} \mid c_{11}c_{22} + c_{12}c_{21} = 1 \right\}.$$

$$|Aut(F_{001})| = |Aut(F_{1\epsilon 1})| = \frac{(q^2-1)(q^2-q)}{q-1} = (q+1)q(q-1)$$

$=$ (the number of ordered triples of distinct points on a conic).

The set of all 0-lines of a conic

Fix an element $\epsilon \in \mathbb{F}_q$ such that $Tr(\epsilon) = 1$.

$$\begin{aligned}\mathcal{L}_0(F_{1\epsilon 1}) &= \{[u,v,w] \mid [u,v,w] \text{ is a 0-line of } F_{1\epsilon 1}\} \\ &= \{[u,v,w] \mid (u,v,w) \in \cup_{\mu\in Tr^{-1}(0)} F_{\epsilon 1\mu}\} \\ &= (\cup_{\mu\in Tr^{-1}(0)} F_{\epsilon 1\mu})^*,\end{aligned}$$

where $S^* = \{[u,v,w] \mid (u,v,w) \in S\}$.

$F_{\epsilon 10} = \{(0,0,1)\}$ means the nucleus and the $\frac{q}{2} - 1$ conics in $\{F_{\epsilon 1\mu} \mid \mu \in Tr^{-1}(0) \setminus \{0\}\}$ are mutually disjoint and $\cup_{\mu\in Tr^{-1}(0)} F_{\epsilon 1\mu}$ forms a maximal $(\frac{q(q-1)}{2}, \frac{q}{2})_q$-arc.

$\mathcal{L}_0(F_{1\epsilon 1})$ is invariant under automorphisms of the conic $F_{1\epsilon 1}$, and itself is a $Aut(F_{1\epsilon 1})$-orbit. Indeed, for any line $\ell_0 \in \mathcal{L}_0(F_{1\epsilon 1})$, the order of $Stab_{\ell_0}(Aut(F_{1\epsilon 1}))$ is $2(q+1)$ and $\frac{|Aut(F_{1\epsilon 1})|}{2(q+1)} = \frac{q(q-1)}{2} = |\mathcal{L}_0(F_{1\epsilon 1})|$.
Thus $\mathcal{K} = \cup_{\mu\in Tr^{-1}(0)} F_{\epsilon 1\mu}$ is expressed as a union of $\frac{q}{2} - 1$ conics and their common nucleus N' for any point $N' \in \mathcal{K}$.

The set of all 0-lines of a conic $F_{ab\lambda}$

Case 1. $Tr(ab) = 1$.

$$\mathcal{L}_0(F_{ab\lambda}) = \{[u,v,w] \mid (u,v,w) \in \cup_{\mu \in \frac{1}{\lambda} Tr^{-1}(0)} F_{ba\mu}\} = (\cup_{\mu \in \frac{1}{\lambda} Tr^{-1}(0)} F_{ba\mu})^*.$$

$F_{ab0} = \{(0,0,1)\}$ means the nucleus and the $\frac{q}{2} - 1$ conics in $\{F_{ba\mu} \mid \mu \in Tr^{-1}(0) \setminus \{0\}\}$ are mutually disjoint and $\mathcal{K}_1 = \cup_{\mu \in Tr^{-1}(0)} F_{ba\mu}$ forms a maximal $(\frac{q(q-1)}{2}, \frac{q}{2})_q$-arc.

Case 2. $Tr(ab) = 0$.

$$\mathcal{L}_0(F_{ab\lambda}) = \{[u,v,w] \mid w \neq 0, (u,v,w) \in \cup_{\mu \in \frac{1}{\lambda} Tr^{-1}(1)} F_{ba\mu}\} = (\cup_{\mu \in \frac{1}{\lambda} Tr^{-1}(1)} F_{ba\mu} \setminus F_{ba0})^*.$$

The $\frac{q}{2}$ conics in $\{F_{ba\mu} \mid \mu \in \frac{1}{\lambda} Tr^{-1}(1)\}$ contain two common points $F_{ba0} = \{P_1, P_2\}$ and $\mathcal{K}_0 = \cup_{\mu \in \frac{1}{\lambda} Tr^{-1}(1)} F_{ba\mu} \setminus \{P_1, P_2\}$ forms a maximal $(\frac{q(q-1)}{2}, \frac{q}{2})_q$-arc.

Since any two conics are projectively equivalent to each other, $\mathcal{K}_0$ and $\mathcal{K}_1$ are also projectively equivalent.

Common 0-lines of conics

Let H be a subspace of $\mathbb{F}_q$ over $\mathbb{F}_2$ with $q = 2^m$.

Let $H^\perp = \{x \in \mathbb{F}_q \mid Tr(\lambda x) = 0 \text{ for all } \lambda \in H\}$.

Then $\dim_{\mathbb{F}_2} H + \dim_{\mathbb{F}_2} H^\perp = \dim_{\mathbb{F}_2} \mathbb{F}_q = m$.

If $Tr(ab) = 1$ then $\mathcal{L}_0(\cup_{\lambda \in H} F_{ab\lambda}) = (\cup_{\mu \in H^\perp} F_{ba\mu})^*$, where $S^* = \{[u, v, w] \mid (u, v, w) \in S\}$.

In general, if J is a subset of $\mathbb{F}_q$, then $\mathcal{L}_0(\cup_{\lambda \in J} F_{ab\lambda}) = \mathcal{L}_0(\cup_{\lambda \in \langle J \rangle} F_{ab\lambda})$, where $\langle J \rangle$ means the subspace spanned by J.

The set of conics disjoint from a line

Consider two projections π_1 and π_2 from the set $\mathcal{H} = \{(C, \ell) \mid C$ is a conic and $C \cap \ell = \emptyset\}$ to the set of conics and the set of lines, respectively.

Then $\left|\pi^{-1}(C)\right| = |\mathcal{L}_0(C)| = \frac{q(q-1)}{2}$

and

$|\mathcal{H}| = (q^5 - q^2) \cdot \frac{q(q-1)}{2}$.

Hence $\left|\pi_2^{-1}(\ell)\right| = \frac{|\mathcal{H}|}{q^2+q+1} = \frac{q^3(q-1)^2}{2}$.

We can count by another way.

Note that $F_{ab\lambda} \cap [0,0,1] = \emptyset$ if and only if $Tr(ab) = 1$. Thus $|\{F_{ab\lambda} \mid F_{ab\lambda} \cap [0,0,1] = \emptyset\}| = (q-1)^2 \cdot \frac{q}{2}$ and the number of points outside the line $[0,0,1]$ is q^2, we also get the number of conics disjoint from a line is $\frac{q^3(q-1)^2}{2}$.

The set of conics $F_{ab\lambda}$ disjoint from F_{001}

Since $DCC(F_{001}, F_{ab\lambda}) = \frac{ab}{(1+\lambda)^2}$, we have

$$\{F_{ab\lambda} \mid F_{001} \cap F_{ab\lambda} = \emptyset\} = \{F_{ab\lambda} \mid Tr(\frac{ab}{(1+\lambda)^2}) = 1\}$$
$$= \{F_{ab\lambda} \mid \lambda \neq 0, 1 \text{ and } \ ab = (1+\lambda)^2\mu \text{ for some } \mu \in Tr^{-1}(1)\}$$
$$= \{F_{ab\lambda} \mid \lambda \neq 0, 1 \text{ and } (a, b, 1+\lambda) \in F_{00\mu} \text{ with } \mu \in Tr^{-1}(1)\}$$

Thus the number of conics $F_{ab\lambda}$ disjoint from F_{001} is $(q-2)\cdot(q+1-2)\cdot\frac{q}{2} = \frac{q(q-1)(q-2)}{2}$.

The set of all conics passing through through given i points

Let m_i ($i \geq 1$) be the number of conics passing given i points, any three of which are noncollinear (if $i \geq 3$). By convention, we let m_0 be the number of all conics. Then we have the following.

$$
\begin{aligned}
m_0 &= q^5 - q^2. \\
m_1 &= q^4 - q^2. \\
m_2 &= q^3 - q^2. \\
m_3 &= (q-1)^2. \\
m_4 &= q - 2. \\
m_5 &= 1. \\
m_6 = \cdots = m_{q+1} &= \begin{cases} 1, & \text{if those } i \text{ points are on a conic.} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

The set of all conics with given points on a conic C_0

Let C_0 be a conic. We find the number of conics which have exactly i points of C_0. For a given i points of C_0, let n_i be the number of conics C such that $C \cap C_0$ is exactly the given points. Then we obtain n_i's as follows.

$$n_{q+1} = 1.$$

$$n_5 = \cdots = n_q = 0.$$

$$n_4 = m_4 - 1 = q - 3.$$

$$n_3 = m_3 - \binom{q-2}{1} n_4 - 1 = 3q - 6.$$

$$n_2 = m_2 - \binom{q-1}{1} n_3 - \binom{q-1}{2} n_4 - 1 = \tfrac{1}{2}(q^3 - 2q^2 + 7q - 8).$$

$$\begin{aligned} n_1 &= m_1 - \binom{q}{1} n_2 - \binom{q}{2} n_3 - \binom{q}{3} n_4 - 1 \\ &= \tfrac{1}{6}(q-1)(2q^3 + 5q^2 - 6q + 6). \end{aligned}$$

$$\begin{aligned} n_0 &= m_0 - \binom{q+1}{1} n_1 - \binom{q+1}{2} n_2 - \binom{q+1}{3} n_3 - \binom{q+1}{4} n_4 - 1 \\ &= \tfrac{1}{8} q \cdot (3q^2 - q + 2)(q-1)^2. \end{aligned}$$

The set of all conics C with $|C \cap C_0| = i$

Let N_i be the number of conics C such that $|C \cap C_0| = i$. Then we obtain the following.

$N_0 = n_0 = \frac{1}{8}q \cdot (3q^2 - q + 2)(q-1)^2$.

$N_1 = \binom{q+1}{1} \cdot n_1 = \frac{1}{6}(q+1)(q-1)(2q^3 + 5q^2 - 6q + 6)$.

$N_2 = \binom{q+1}{2} \cdot n_2 = \frac{1}{4}(q+1)q(q^3 - 2q^2 + 7q - 8)$.

$N_3 = \binom{q+1}{3} \cdot n_3 = \frac{1}{2}(q+1)q(q-1)(q-2)$.

$N_4 = \binom{q+1}{4} \cdot n_4 = \frac{1}{24}(q+1)q(q-1)(q-2)(q-3)$.

$N_5 = \cdots = N_q = 0$.

$N_{q+1} = 1$.

Remark. The number of conics disjoint from a given conic is $N_0 = \frac{1}{8}q \cdot (3q^2 - q + 2)(q-1)^2$.

References

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Thank you for your attention!!!