Linear Coded Caching Schemes

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1. Background
2. Coded caching scheme
3. Linear constructions

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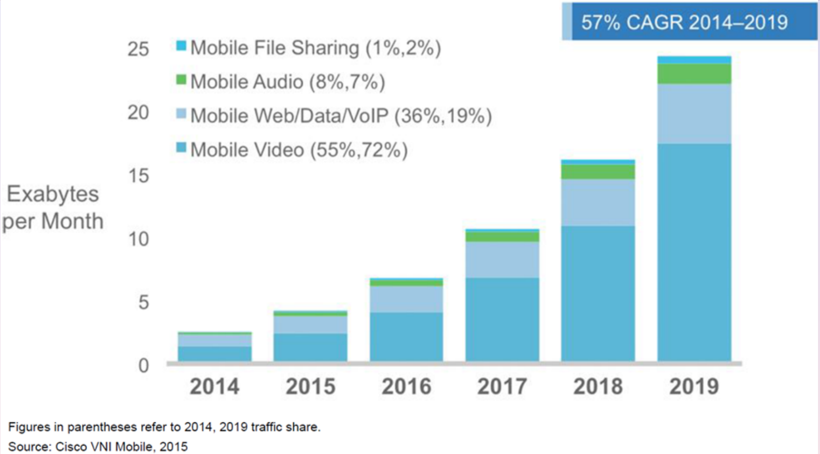
The amount of transmission in wireless network



- Video-on-demand drives wireless traffic growth
- The available bandwidth is finite

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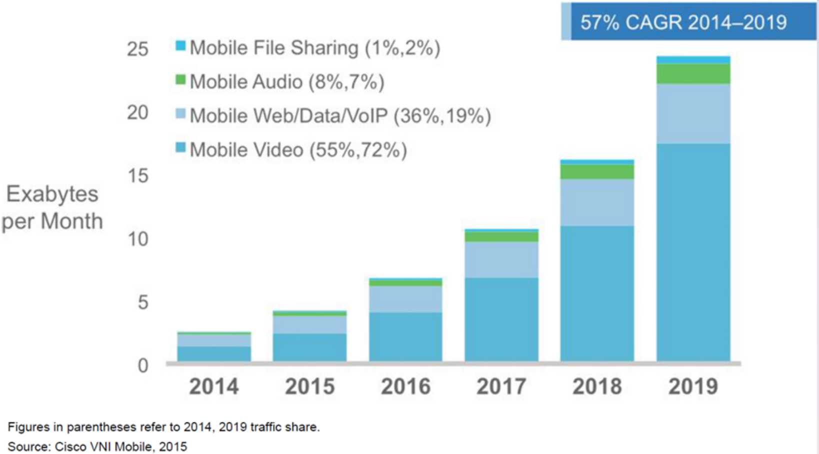

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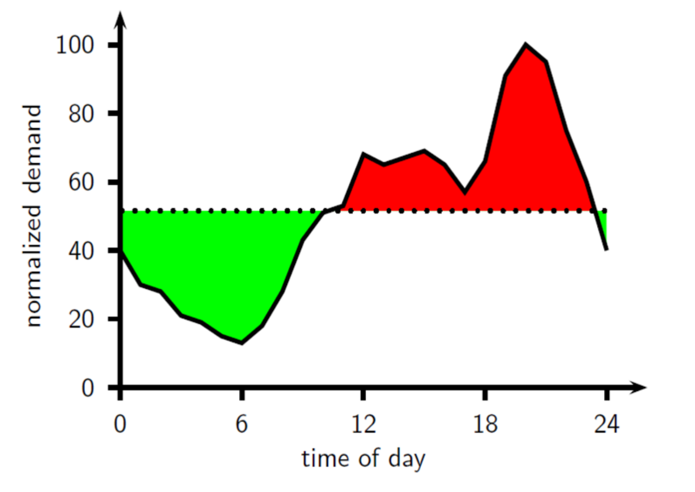

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- Peak traffic times: $12 \sim 24$h

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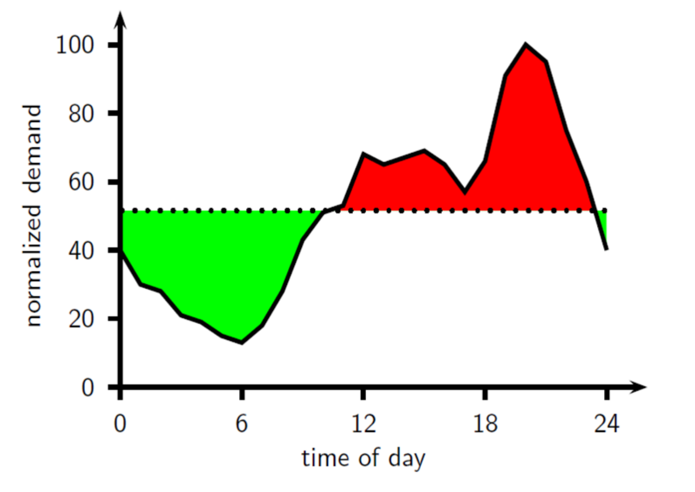

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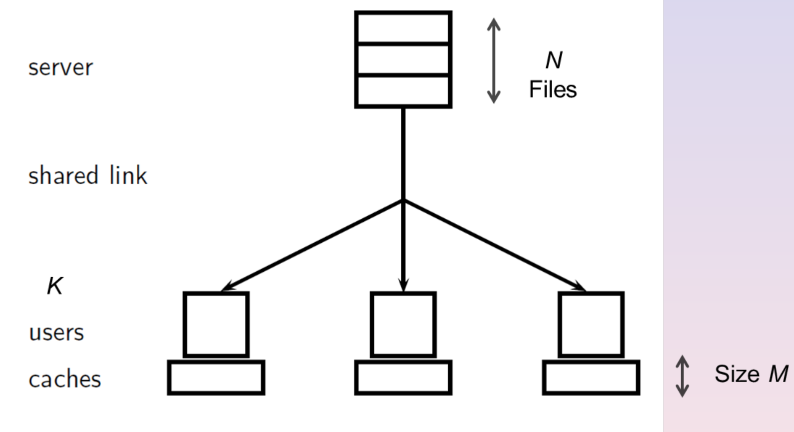
Centralized wireless network



- A server stores N files, each of size F packets ($N = 3$)
- K users, each access a cache of size MF packets ($K = 3$)

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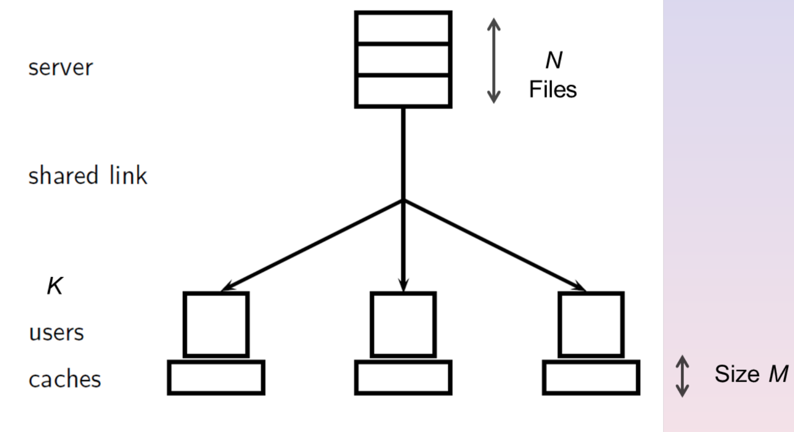

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Caching system

- Placement Phase
  - parts of each file is partially cached at each user
  - without the knowledge of user's demands
- Delivery Phase

  requested file with help its cached contents.

$$R = \max\left\{ R_{\mathbf{d}} \mid \forall \mathbf{d} \in [0, N)^K \right\}.$$

  R is always called the rate of a coded caching scheme.
- **Objective 1:** R is as small as possible.

Caching system

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MN scheme

In 2014, Ali and Niesen proposed that

coded caching scheme can be used to further reduce R.

- The rate of MN scheme is ordered optimal
- It is widely used in heterogeneous wireless network, such as D2D, hierarchical network and so on.
- Best paper award of IEEE IT
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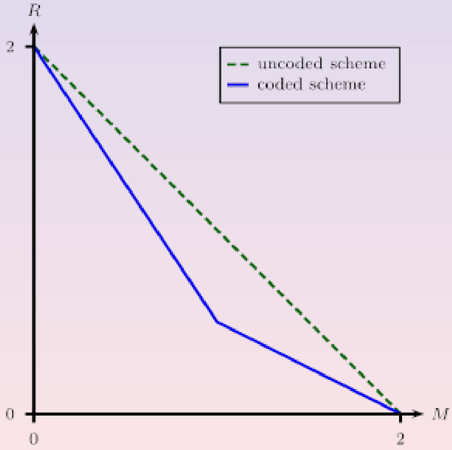
Comparison: Uncoded VS Coded

N files, $K = N$ users, Cache size M files



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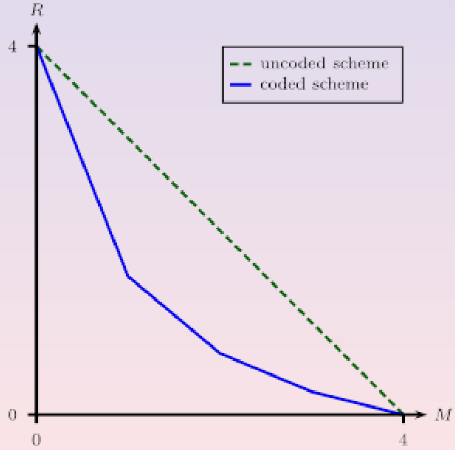

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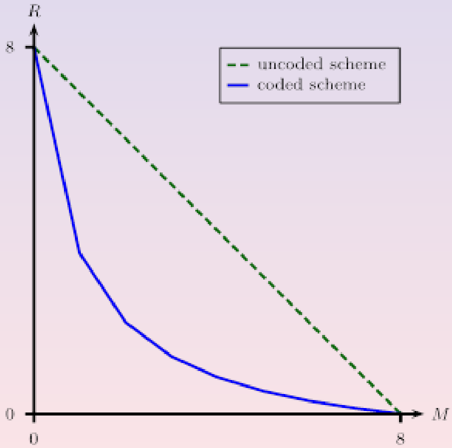

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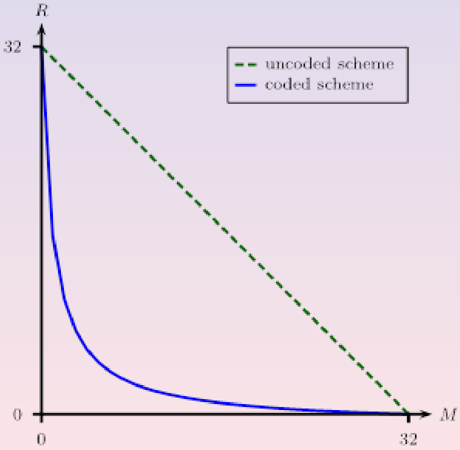

The second objective

- **Disadvantage:** MN scheme is unfeasible when K is large.
- **Objective 2:** F is as small as possible for the fixed R.

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Table: Previously known constructions

results	M/N	R	F
MN scheme	$\frac{1}{q}$	$\frac{K(q-1)}{q+K}$	$\sim \frac{q}{\sqrt{2\pi K(q-1)}} \cdot e^{\frac{K}{q}\left(\ln q+(q-1)\ln \frac{q}{q-1}\right)}$
The scheme	$\frac{1}{q}$	$q-1$	$e^{(\frac{K}{q}-1)\ln q}$

[3] Q. Yan, M. Cheng, X. Tang, and Q. Chen, IEEE Trans. Inform. Theory, 63(9), 2017.

Placement phase:

- User k caches

$$\mathcal{Z}_k = \{\phi_k(W_n) \mid n \in [0, N)\}$$

where $\phi_k(x)$ (*caching function*) has the size of $\frac{FM}{N}$ packets.

Delivery phase:

- Given a request $\mathbf{d} = (d_0, d_1, \cdots, d_{K-1})$, sever sends

$$\mathbf{X_d} = \psi_0(W_{d_0}) + \psi_1(W_{d_1}) + \ldots + \psi_{K-1}(W_{d_{K-1}})$$

to users where ψ_k(*coding function*) has the size of RF packets.

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Linear characterization

- User k obtains
$$W'_{d_k} = \chi_k(\mathbf{X_d})$$
where χ_k(*decoding function*) has the size of $F(1-\frac{M}{N})$ packets.
- Our goal: W_{d_k} can be obtained by $\phi_k(W_{d_k})$ and W'_{d_k}.

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- Placement phase: User k caches

$$\mathcal{Z}_k = \{\mathbf{S}_k W_n \mid n \in [0, N)\}$$

where $\mathbf{S}_i$ (*caching matrix*) is an $\frac{FM}{N} \times F$ matrix.

- Delivery phase: Given a request $\mathbf{d} = (d_0, d_1, \cdots, d_{K-1})$, sever sends

$$\mathbf{X}_\mathbf{d} = \mathbf{A}_0 W_{d_0} + \mathbf{A}_1 W_{d_1} + \ldots + \mathbf{A}_{K-1} W_{d_{K-1}}$$

to users where $\mathbf{A}_k$(*coding matrix*) is a $RF \times F$ matrix.

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$$W'_{d_k} = \mathbf{S}'_k \mathbf{X}_\mathbf{d}$$

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Theorem

For any request $\mathbf{d}$ and the related signals $\mathbf{X}_{\mathbf{d}}$, user k can obtain the required W_{d_k} if and only if the matrices satisfy the following conditions.

$$\operatorname{rank}\begin{pmatrix} \mathbf{S}_k \\ \mathbf{S}'_k \mathbf{A}_{k'} \end{pmatrix} = \begin{cases} F, & \text{if } \ k = k' \\ FM/N, & \text{otherwise} \end{cases} \tag{1}$$

for all $0 \leq k' < K$.

Remark

- Formula (1) is exactly one condition of an minimum storage regenerating code

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- Minimum storage regenerating (MSR) code was introduced in [4] for distributed storage systems.
- Assume that a file of size $\mathcal{M} = k\alpha$ denoted by the column vector $\mathbf{W} \in \mathbf{F}_p^{k\alpha}$ is partitioned in k parts

$$\mathbf{W} = \{\mathbf{W}_0, \mathbf{W}_1, \cdots, \mathbf{W}_{k-1}\}$$

each of size α, where p is a primepower.
- We encode $\mathbf{W}$ using an $(n = k + r, k)$ MDS code and store it across k systematic and r parity storage nodes.

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Theorem

If an $(n = K + r, K)$ MSR code has optimal repairing bandwidth, then a (K, M, N) caching scheme with

$$F = \alpha \qquad \frac{M}{N} = \frac{1}{r} \qquad R = r - 1$$

can be obtained.

(1) holds if and only if the following formula holds.

$$\begin{cases} \mathbf{S}'_k \mathbf{A}_{k'} \subseteq \mathbf{S}_k & \text{if} \quad k \neq k' \\ \mathbf{S}_k + \mathbf{S}'_k \mathbf{A}_{k'} = \mathbb{F}_2^F & \text{if} \quad k = k' \end{cases} \qquad k, k' \in [1, K)$$

Here the sum of two subspace $\mathcal{U}$, $\mathcal{V}$ of $\mathbb{F}^F$ is defined as

$$\mathcal{U} + \mathcal{V} = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in \mathcal{U}, \mathbf{v} \in \mathcal{V}\}.$$

Notations

- Given an integer $s \in [0, q^m)$ where $m \in \mathbb{N}^+$, with

$$s = \sum_{l=0}^{m-1} s_l q^l$$

for integers $s_l \in [0, q)$, we refer to

$$s = (s_{m-1}, \cdots, s_0)_q$$

as the *q-ary representation* of s.

- For each integer s let

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Notations

- There are m partitions of $[0, q^m)$, *i.e.*, for $u = 0, 1, \cdots, m-1$,

$$\mathcal{V}_{u,v} = \{(s_{n-1}, \cdots, s_0)_q \mid s_u = v\}, \ 0 \leq v < q.$$

Example ($q = 3$, $m = 2$)

$$\mathcal{V}_{0,0} = \{(0,0),\ (1,0),\ (2,0)\} = \{0,\ 3,\ 6\}$$
$$\mathcal{V}_{0,1} = \{(0,1),\ (1,1),\ (2,1)\} = \{1,\ 4,\ 7\}$$
$$\mathcal{V}_{0,2} = \{(0,2),\ (1,2),\ (2,2)\} = \{2,\ 5,\ 8\}$$
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\end{aligned}$$

- For each u and v, define

$$\mathcal{E}_{u,v} = \{e_s \mid s \in \mathcal{V}_{u,v}\}$$

and

$$\mathcal{Q}_u = \left\{ \sum_{s_u=0}^{q-1} e_{(s_{m-1},\cdots,s_0)} \;\middle|\; s_j \in \{0,1\}, j \neq u \right\}$$

where the sum is performed under modulo q.

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$$\mathcal{E}_{0,0} = \{\mathbf{e}_0,\ \mathbf{e}_3,\ \mathbf{e}_6\} \quad \mathcal{E}_{0,1} = \{\mathbf{e}_1,\ \mathbf{e}_4,\ \mathbf{e}_7\} \quad \mathcal{E}_{0,2} = \{\mathbf{e}_2,\ \mathbf{e}_5,\ \mathbf{e}_8\}$$
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$$\mathcal{Q}_0 = \{\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5,\ \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8\},$$
$$\mathcal{Q}_1 = \{\mathbf{e}_0 + \mathbf{e}_3 + \mathbf{e}_6,\ \mathbf{e}_1 + \mathbf{e}_4 + \mathbf{e}_7,\ \mathbf{e}_2 + \mathbf{e}_5 + \mathbf{e}_8\}.$$

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Example ($q = 3$, $m = 2$)

$$\mathcal{E}_{0,0} = \{\mathbf{e}_0,\ \mathbf{e}_3,\ \mathbf{e}_6\} \quad \mathcal{E}_{0,1} = \{\mathbf{e}_1,\ \mathbf{e}_4,\ \mathbf{e}_7\} \quad \mathcal{E}_{0,2} = \{\mathbf{e}_2,\ \mathbf{e}_5,\ \mathbf{e}_8\}$$
$$\mathcal{E}_{1,0} = \{\mathbf{e}_0,\ \mathbf{e}_1,\ \mathbf{e}_2\} \quad \mathcal{E}_{1,1} = \{\mathbf{e}_3,\ \mathbf{e}_4,\ \mathbf{e}_5\} \quad \mathcal{E}_{1,2} = \{\mathbf{e}_6,\ \mathbf{e}_7,\ \mathbf{e}_8\}$$

$$\mathcal{Q}_0 = \{\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5,\ \mathbf{e}_6 + \mathbf{e}_7 + \mathbf{e}_8\},$$
$$\mathcal{Q}_1 = \{\mathbf{e}_0 + \mathbf{e}_3 + \mathbf{e}_6,\ \mathbf{e}_1 + \mathbf{e}_4 + \mathbf{e}_7,\ \mathbf{e}_2 + \mathbf{e}_5 + \mathbf{e}_8\}.$$

Notations

- $$\phi_{u,v}(s) = \begin{cases} 1 & \text{if } s \in \mathcal{V}_{u,v} \\ 0 & \text{otherwise} \end{cases}$$
$$\varphi_{u,v}(s) = (s_0, \ldots, s_{u-1}, v, s_{u+1}, \ldots, s_{m-1}).$$

- $$\mathbf{C}_{u,v,v'} = \begin{pmatrix} \phi_{u,v}(0)\mathbf{e}_{\varphi_{u,v'}(0)} \\ \phi_{u,v}(1)\mathbf{e}_{\varphi_{u,v'}(1)} \\ \cdots \\ \phi_{u,v}(q^m-1)\mathbf{e}_{\varphi_{u,v'}(q^m-1)} \end{pmatrix} + \begin{pmatrix} \phi_{u,v'}(0)\mathbf{e}_0 \\ \phi_{u,v'}(1)\mathbf{e}_1 \\ \cdots \\ \phi_{u,v'}(q^m-1)\mathbf{e}_{q^m-1} \end{pmatrix}$$

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Example ($q = 3,\ m = 2$)

$$\mathbf{C}_{0,0,1} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_1 \\ 0 \\ \mathbf{e}_4 \\ \mathbf{e}_4 \\ 0 \\ \mathbf{e}_7 \\ \mathbf{e}_7 \\ 0 \end{pmatrix}, \mathbf{C}_{0,0,2} = \begin{pmatrix} \mathbf{e}_2 \\ 0 \\ \mathbf{e}_2 \\ \mathbf{e}_5 \\ 0 \\ \mathbf{e}_5 \\ \mathbf{e}_8 \\ 0 \\ \mathbf{e}_8 \end{pmatrix}, \mathbf{C}_{0,1,0} = \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_0 \\ 0 \\ \mathbf{e}_3 \\ \mathbf{e}_3 \\ 0 \\ \mathbf{e}_6 \\ \mathbf{e}_6 \\ 0 \end{pmatrix}, \mathbf{C}_{1,1,2} = \begin{pmatrix} 0 \\ \mathbf{e}_2 \\ \mathbf{e}_2 \\ 0 \\ \mathbf{e}_5 \\ \mathbf{e}_5 \\ 0 \\ \mathbf{e}_8 \\ \mathbf{e}_8 \end{pmatrix},$$

$$\mathbf{C}_{0,2,0} = \begin{pmatrix} \mathbf{e}_0 \\ 0 \\ \mathbf{e}_0 \\ \mathbf{e}_3 \\ 0 \\ \mathbf{e}_3 \\ \mathbf{e}_6 \\ 0 \\ \mathbf{e}_6 \end{pmatrix}, \mathbf{C}_{0,2,1} = \begin{pmatrix} 0 \\ \mathbf{e}_1 \\ \mathbf{e}_1 \\ 0 \\ \mathbf{e}_4 \\ \mathbf{e}_4 \\ 0 \\ \mathbf{e}_7 \\ \mathbf{e}_7 \end{pmatrix}, \mathbf{C}_{0,3,0} = \begin{pmatrix} \mathbf{e}_0 \\ 0 \\ 0 \\ \mathbf{e}_3 \\ 0 \\ 0 \\ \mathbf{e}_6 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_{0,3,1} = \begin{pmatrix} 0 \\ \mathbf{e}_1 \\ 0 \\ 0 \\ \mathbf{e}_4 \\ 0 \\ 0 \\ \mathbf{e}_7 \\ 0 \end{pmatrix}$$

- Caching matrices

$$\mathbf{S}_{u,v} = \mathcal{E}_{u,v}, \mathbf{S}_{u,q} = \mathcal{Q}_u$$

- Coding matrices:

$$\mathbf{A}_{u,v} = \begin{pmatrix} \mathbf{C}_{u,v,v_0} \\ \mathbf{C}_{u,v,v_1} \\ \vdots \\ \mathbf{C}_{u,v,v_{q-1}} \end{pmatrix}, v_i \in [0,q) \setminus \{v\} \text{ and } \mathbf{A}_{u,q} = \begin{pmatrix} \mathbf{C}_{u,q,0} \\ \mathbf{C}_{u,q,1} \\ \vdots \\ \mathbf{C}_{u,q,q-2} \end{pmatrix}$$

- Decoding matrices

$$\mathbf{S}'_{u,v} = \begin{pmatrix} \mathbf{E}_{u,v} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{E}_{u,v} \end{pmatrix}$$

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Table: Comparisons

results	M/N	R	F
MN scheme	$\frac{1}{q}$	$\frac{K(q-1)}{q+K}$	$\sim \frac{q}{\sqrt{2\pi K(q-1)}} \cdot e^{\frac{K}{q}\left(\ln q+(q-1)\ln\frac{q}{q-1}\right)}$
Known scheme	$\frac{1}{q}$	$q-1$	$e^{(\frac{K}{q}-1)\ln q}$
New Scheme	$\frac{1}{q}$	$q-1$	$e^{\frac{K}{q+1}\ln q}$

Theorem

For any positive integers q, z, m with $q \geq 2$ and $z < q$,there exists a coded caching scheme with parameters

$$K = m(q+1)\left\lfloor \frac{q-1}{q-z} \right\rfloor, \quad \frac{M}{N} = \frac{z}{q}, \quad R = q - z, \quad F = q^m.$$

The operation is over the finite filed $\mathbb{F}_2$.

- We proposed a new viewpoint to study coded caching scheme and constructed a new class of coded caching schemes over $\mathbb{F}_2$.

- For the fixed parameters K, $\frac{M}{N}$ and R, could we further reduce the value of F by increasing the size of filed?

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Thank You!!!