On the structure of the non-full-rank STS

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(joint work with M. Shi and L. Xu)

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Steiner triple systems (STS)

A Steiner triple system STS(v) is a finite set S of cardinality v whose elements are called points, provided with a collection of 3-subsets called blocks such that every 2-subset of S is contained in one and only one block. We assume that $S = \{1, \ldots, v\}$ and do not distinguish a block b with its characteristic vectors, that is, the v-tuple of 0s and 1s having 1s in the coordinates numbered by the elements of b. E.g., $(0, 1, 0, 1, 1, 0, 0) = \{2, 4, 5\}$ $(v = 7)$.

p-rank of STS

The dimension of the space over $\mathrm{GF}(p)$ spanned by the blocks (to be exact, by their characteristic vectors) of a Steiner triple system T is called the p-rank of T. As shown in [1], the p-rank of every STS(v) is v for all prime p except 2 and 3.

[1] J. Doyen, X. Hubaut, and M. Vandensavel. Ranks of incidence matrices of Steiner triple systems. *Mathematische Zeitschrift*, 163(3):251–259, Oct. 1978. DOI: 10.1007/BF01174898

A series of papers are devoted to the study of STS(v) of deficient 2-rank. In particular, in [2], [3], [4], there found formulas for the number of STS($2^m - 1$) of 2-rank $2^m - m$, $2^m - m + 1$, $2^m - m + 2$, respectively.

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[2] V. D. Tonchev. A mass formula for Steiner triple systems STS($2^n - 1$) of 2-rank $2^n - n$. *J. Comb. Theory, Ser. A*, 95(2):197–208, 2001. DOI: 10.1006/jcta.2000.3161

[3] V. A. Zinoviev and D. V. Zinoviev. Remark on "Steiner triple systems $S(2^m - 1, 3, 2)$ of rank $2^m - m + 1$ over F_2" published in ... *Probl. Inf. Transm.*, 49(2):192–195, 2013. DOI: 10.1134/S0032946013020087

[4] D. V. Zinoviev. On the number of Steiner triple systems $S(2^m - 1, 3, 2)$ of rank $2^m - m + 2$ over F_2. *Discrete Math.*, 339(11):2727–2736, 2016. Corrected formula: `https://arxiv.org/abs/1512.00187`

In some recent papers, not only 2-rank, but also 3-rank of STS is considered.
[5]
[6]
In [7], a formula for the number of STS orthogonal to a fixed binary or ternary subspace was found.
In [8], we count the number of all STS(v) of a prescribed non-full 2- or 3-rank.

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[5] D. Jungnickel and V. D. Tonchev. On Bonisoli's theorem and the block codes of Steiner triple systems. *Des. Codes Cryptography*, 86(3):449–462, 2018.

[6] D. Jungnickel, S. S. Magliveras, V. D. Tonchev, and A. Wassermann. The classification of Steiner triple systems on 27 points with 3-rank 24. *Des. Codes Cryptography*, 2018.

[7] D. Jungnickel and V. D. Tonchev. Counting Steiner triple systems with classical parameters and prescribed rank. *J. Comb. Theory, Ser. A*, 2018. To appear.

[8] M.Shi, L.Xu, D.Krotov. The number of the non-full-rank Steiner triple systems. Submitted. 2018

Structure of orthogonal subspace, GF(3)

Let $v \equiv 1, 3 \bmod 6$; that is, there exist STS(v). By V^v, we denote the vector space of all v-tuples over $\mathrm{GF}(3)$. Denote by $\mathcal{D}$ the set of subspaces of V^v, each including the all-one vector and being orthogonal to at least one STS(v); denote

$$\mathcal{D}_i = \{D \in \mathcal{D} : \dim(D) = i + 1\}.$$

Lemma (variant of [Doyen, Hubaut, Vandensavel, 1978])

Let M be a $(i+1) \times v$ generator matrix for $D \in \mathcal{D}_i$, and let the first row of M be the all-one vector. Then M consists of 3^i different columns, each occurring $v/3^i$ times.

$$\begin{pmatrix} 111111111111111111111111111 \\ 000000000111111111222222222 \\ 000111222000111222000111222 \end{pmatrix}.$$

Structure of orthogonal subspace. PROOF (begin)

Lemma

Let M be a $(i+1) \times v$ generator matrix for $D \in \mathcal{D}_i$, and let the first row of M be the all-one vector. Then M consists of 3^i different columns, each occurring $v/3^i$ times.

- Without loss of generality, the 1st column is $(1, 0, ..., 0)^{\mathrm{T}}$.
- (*) *If a and b are columns of M, then $-a-b$ is.*
- (**) *If c, d are columns of M, then $c + d - (1, 0, ..., 0)^{\mathrm{T}}$ is.*
- So, the set of columns of the matrix M' obtained from M by removing the first row is closed under the addition. So, all possible 3^i columns occurs.

$$\begin{matrix}
1&1&1&1&1&1&1&\mathbf{1}&1&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{1}&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&\mathbf{0}&0&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{2}&2&2&2&2&2&2&2&2\\
0&0&0&1&1&1&2&\mathbf{2}&2&0&0&0&1&1&\mathbf{1}&2&2&2&\mathbf{0}&0&0&1&1&1&2&2&2\\
&&&&&&&1&&&&&&&1&&&&1&&&&&&&&
\end{matrix}$$

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$$
\begin{array}{ccccccccccccccccccccccccccc}
1&1&1&1&1&1&1&\mathbf{1}&1&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{1}&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&\mathbf{0}&0&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{2}&2&2&2&2&2&2&2&2\\
0&0&0&1&1&1&2&\mathbf{2}&2&0&0&0&1&1&\mathbf{1}&2&2&2&\mathbf{0}&0&0&1&1&1&2&2&2\\
&&&&&&&1&&&&&&&1&&&&1&&&&&&&&
\end{array}
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$$
\begin{array}{ccccccccccccccccccccccccccc}
1&1&1&1&1&1&1&\mathbf{1}&1&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{1}&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&\mathbf{0}&0&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{2}&2&2&2&2&2&2&2&2\\
0&0&0&1&1&1&2&\mathbf{2}&2&0&0&0&1&1&\mathbf{1}&2&2&2&\mathbf{0}&0&0&1&1&1&2&2&2\\
&&&&&&&1&&&&&&&1&&&&1&&&&&&&&
\end{array}
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\begin{array}{ccccccccccccccccccccccccccc}
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0&0&0&0&0&0&0&\mathbf{0}&0&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{2}&2&2&2&2&2&2&2&2\\
0&0&0&1&1&1&2&\mathbf{2}&2&0&0&0&1&1&\mathbf{1}&2&2&2&\mathbf{0}&0&0&1&1&1&2&2&2\\
&&&&&&&1&&&&&&&1&&&&1&&&&&&&&
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1&1&1&1&1&1&1&\mathbf{1}&1&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{1}&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&\mathbf{0}&0&1&1&1&1&1&\mathbf{1}&1&1&1&\mathbf{2}&2&2&2&2&2&2&2&2\\
0&0&0&1&1&1&2&\mathbf{2}&2&0&0&0&1&1&\mathbf{1}&2&2&2&\mathbf{0}&0&0&1&1&1&2&2&2\\
&&&&&&&1&&&&&&&1&&&&1&&&&&&&&
\end{array}
$$

Structure of orthogonal subspace. PROOF (end)

Lemma

Let M be a $(i+1) \times v$ generator matrix for $D \in \mathcal{D}_i$, and let the first row of M be the all-one vector. Then M consists of 3^i different columns, each occurring $v/3^i$ times.

- It remains to prove that all groups are of the same size.
- For given two different columns a and b, we fix an element (position) corresponding to the column $-a-b$, and count the number of triples.

$$
\begin{array}{l}
\ldots 1111111 \ldots 1111111 \ldots 1111111 \ldots \\
\ldots 0000000 \ldots 1111111 \ldots 2222222 \ldots \\
\ldots 2222222 \ldots 1111111 \ldots 0000000 \ldots \\
\quad \mathbf{1} \qquad\qquad\qquad\qquad 1 \qquad\quad 1
\end{array}
$$

- The proof is over.

Structure of orthogonal subspace. PROOF (end)

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$$\begin{array}{l} \ldots 1111111 \ldots 1111111 \ldots 1111111 \ldots \\ \ldots 0000000 \ldots 1111111 \ldots 2222222 \ldots \\ \ldots 2222222 \ldots 1111111 \ldots 0000000 \ldots \\ \quad\;\; \mathbf{1} \qquad\qquad\qquad\qquad 1 \qquad\qquad 1 \end{array}$$

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$$\begin{array}{c}
\ldots 1111111 \ldots 1111111 \ldots 1111111 \ldots \\
\ldots 0000000 \ldots 1111111 \ldots 2222222 \ldots \\
\ldots 2222222 \ldots 1111111 \ldots 0000000 \ldots \\
\mathbf{1} \qquad\qquad\qquad\qquad 1 \qquad\qquad 1
\end{array}$$

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$$
\begin{array}{l}
\ldots 1\,1\,1\,1\,1\,1\,1 \ldots 1\,1\,1\,1\,1\,1\,1 \ldots 1\,1\,1\,1\,1\,1\,1 \ldots \\
\ldots 0\,0\,0\,0\,0\,0\,0 \ldots 1\,1\,1\,1\,1\,1\,1 \ldots 2\,2\,2\,2\,2\,2\,2 \ldots \\
\ldots 2\,2\,2\,2\,2\,2\,2 \ldots 1\,1\,1\,1\,1\,1\,1 \ldots 0\,0\,0\,0\,0\,0\,0 \ldots \\
\quad\;\; \mathbf{1} \qquad\qquad\qquad\qquad\quad 1 \qquad\qquad 1
\end{array}
$$

- The proof is over.

The structure of non-full-3-rank STS

Lemma (variant of [Jungnickel and Tonchev, 2018+])

Given a subspace D from $\mathcal{D}_j$, the set of $STS(v)$ orthogonal to D is in one-to-one correspondence with the collections of 3^j Steiner triple systems of order $v/3^j$ and $3^j(3^j-1)/6$ latin squares of order $v/3^j$.

- For each of 3^j groups, the triples with all 3 points in these group form STS($v/3^j$).

```
...1111111...
...0000000...
...2222222...
   111
   1  11
   1    11
    1 1 1
    1  1 1
     11  1
```

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Given a subspace D from $\mathcal{D}_j$, the set of $STS(v)$ orthogonal to D is in one-to-one correspondence with the collections of 3^j Steiner triple systems of order $v/3^j$ and $3^j(3^j-1)/6$ latin squares of order $v/3^j$.

- For each of 3^j groups, the triples with all 3 points in these group form $\mathrm{STS}(v/3^j)$.

$$
\begin{array}{ccccccccc}
\ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \ldots \\
 & 1 & 1 & 1 & & & & & \\
 & 1 & & & 1 & 1 & & & \\
 & 1 & & & & & 1 & 1 & \\
 & & 1 & & 1 & & 1 & & \\
 & & 1 & & & 1 & & 1 & \\
 & & & 1 & 1 & & & 1 &
\end{array}
$$

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- For every 3 distinct groups $\{\alpha_1,\ldots,\alpha_{v/3^j}\}$, $\{\beta_1,\ldots,\beta_{v/3^j}\}$, $\{\gamma_1,\ldots,\gamma_{v/3^j}\}$ corresponding to columns a, b, c with $a+b+c=0$, the triples with one point in each of these 3 groups have the form $\{\alpha_x,\beta_y,\gamma_{f(x,y)}\}$ for some latin square f of order $v/3^j$.

```
...1111111...1111111...1111111...
...0000000...1111111...2222222...
...2222222...1111111...0000000...
   1              1              1
   1                 1                 1
    1                1              1
```

0	1	2	3	4	5	6
2	3	4	5	6	0	1
5	6	0	1	2	3	4
6	0	1	2	3	4	5
3	4	5	6	0	1	2
4	5	6	0	1	2	3
1	2	3	4	5	6	0

The number of STS with prescribed orthogonal space

Corollary

Given a subspace D from $\mathcal{D}_j$, the number $\Phi(D)$ of $STS(v)$ orthogonal to D equals $\Phi_{v,j}$, where

$$\Phi_{v,j} = \Psi_{v/3^j}^{3^j} \Lambda_{v/3^j}^{3^j(3^j-1)/6},$$

Ψ_u is the number of $STS(u)$, and Λ_u is the number of latin squares of order u.

orthogonal $\rightarrow$ dual?

- Denote by $N(D)$ the number of STS whose dual is D.
- Given a subspace D from $\mathcal{D}_j$, we know the number of STS which are orthogonal to D.
- That is to say, we know $\sum_{D' \supset D} N(D)$.
- How to find $N(D)$?

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- Denote by $N(D)$ the number of STS whose dual is D.
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- How to find $N(D)$?

the Transform

- Fourier transform?
- larva-nymph transform?
- Möbius transform?
- boson-fermion transform?

Möbius transform

Lemma

Assume that v is divided by 3^k and k is the largest integer with this property. Let $i \in \{0, \ldots, k\}$, and let D be in $\mathcal{D}_i$. The number of $STS(v)$ with dual space D equals $\Upsilon_{v,i}$, where

$$\Upsilon_{v,i} = \sum_{j=i}^{k} \Gamma_{v,i,j} \mu^{(3)}_{j-i} \Phi_{v,j}, \tag{1}$$

where

- *every subspace from $\mathcal{D}_i$ is contained in exactly $\Gamma_{v,i,j}$ subspaces from $\mathcal{D}_j$,*
- $\mu^{(q)}_i = (-1)^i q^{\binom{i}{2}}$ *(Möbius coefficients),*
- $\Phi_{v,j}$ *are from Corollary above*

Main theorem, GF(3)

Theorem

Assume that v is divided by 3^k and k is the largest integer with this property. Let $i \in \{0, \ldots, k\}$. The total number of different $STS(v)$ of 3-rank $v - i - 1$ equals

$$\Gamma_{v,0,i} \sum_{j=i}^{k} \Gamma_{v,i,j} \mu_{j-i}^{(3)} \Phi_{v,j}, \qquad \textit{where } \mu_l^{(3)} = (-1)^l 3^{\binom{l}{2}}, \quad \Phi_{v,j} = \Psi_{v/3^j}^{3^j} \Lambda_{v/3^j}^{3^j(3^j}$$

$$\Gamma_{v,i,j} = \left(\frac{v}{3^i}!\right)^{3^i} \Big/ 3^{\frac{(j+i+1)(j-i)}{2}} \left(\frac{v}{3^j}!\right)^{3^j} \prod_{s=1}^{j-i} (3^s - 1),$$

Ψ_u is the number of $STS(u)$ (and also the number of idempotent totally symmetric latin squares of order u), and Λ_u is the number of latin squares of order u.

one of partial cases: 3-rank +2

Corollary

The number of $STS(v)$, $v = 3^k$, *of* 3*-rank* $v - k + 1$ *is*

$$\frac{v!}{2^{k+2} \cdot 3^{\frac{k(k+1)}{2}-1} \cdot [k-2]_3!}$$

$$\times \left(\frac{(2^{35} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 5231 \cdot 3824477)^{\frac{v(v-9)}{486}}}{2^{4v/9-4} \cdot 3^{v/3-2k+1}} \right.$$

$$\left. - 2^{v^2/27-4v/9+3} \cdot 3^{v^2/54-7v/18+k-1} + 1 \right).$$

Now, consider 2-rank (over GF(2))

Structure of orthogonal subspace, GF(2)

Let $w+1 \equiv 1, 3 \bmod 6$; that is, there exist STS$(w+1)$. By V^{w+1}, we denote the vector space of all $(w+1)$-tuples over $\mathrm{GF}(2)$. Denote by $\mathcal{D}$ the set of subspaces of V^{w+1} orthogonal to at least one STS(v); denote

$$\mathcal{D}_i = \{D \in \mathcal{D} : \dim(D) = i\}.$$

Lemma (variant of [Doyen, Hubaut, Vandensavel, 1978])

Let M be a $i \times (w-1)$ generator matrix for $D \in \dot{\mathcal{D}}_i$. Then each of the $2^i - 1$ nonzero columns of height i occurs $w/2^i$ times as a column of M, while the all-zero column occurs $w/2^i - 1$ times.

$$\left(\begin{array}{ccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1
\end{array}\right).$$

Structure of orthogonal subspace, PROOF, GF(2)

- (*) *If a and b are different nonzero columns of M, then $a + b$ is also a column of M.* The proof is similar to the ternary case.
- Since the rank of M is i, it contains i linearly independent columns. It follows from (*) that it contains all $2^i - 1$ different nonzero columns of height i.
- It remains to show that the all-zero column occurs one less times than every other column.

$$
\begin{array}{ccc}
\ldots 0000000 \ldots & \mathbf{1} & 1111111 \ldots \\
\ldots 0000000 \ldots & \mathbf{0} & 0000000 \ldots \\
\ldots 0000000 \ldots & \mathbf{1} & 1111111 \ldots \\
1 & \mathbf{1} & 1
\end{array}
$$

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$$
\begin{array}{ccc}
\ldots 0000000 \ldots & \mathbf{1}1111111 \ldots \\
\ldots 0000000 \ldots & \mathbf{0}0000000 \ldots \\
\ldots 0000000 \ldots & \mathbf{1}1111111 \ldots \\
1 & \mathbf{1} \qquad 1
\end{array}
$$

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- It remains to show that the all-zero column occurs one less times than every other column.

$$
\begin{array}{ccc}
\ldots 0\,0\,0\,0\,0\,0\,0 \ldots & \mathbf{1} & 1\,1\,1\,1\,1\,1\,1 \ldots \\
\ldots 0\,0\,0\,0\,0\,0\,0 \ldots & \mathbf{0} & 0\,0\,0\,0\,0\,0\,0 \ldots \\
\ldots 0\,0\,0\,0\,0\,0\,0 \ldots & \mathbf{1} & 1\,1\,1\,1\,1\,1\,1 \ldots \\
1 & \mathbf{1} & 1
\end{array}
$$

The structure of non-full-2-rank STS

Lemma (variant of [Jungnickel and Tonchev, 2018+])

Given a subspace D from $\dot{\mathcal{D}}_j$, the set of $STS(w-1)$ orthogonal to D is in one-to-one correspondence with the collections of one $STS(w/2^j-1)$, 2^j-1 symmetric latin squares of order $w/2^j-1$, and $(2^j-1)(2^j-2)/6$ latin squares of order $w/2^j$.

- triples with 3 ones in the "all-zero-column group" form to $\mathrm{STS}(w/2^j-1)$;
- for different columns a, b, c with $a+b+c=0$ we have a latin square (similarly to the 3-ary case);
- for non-zero column a the triples with two points in the corresponding group of coordinates and one point in the "all-zero group" form a symmetric square of order $w/2^j$.

The structure of non-full-2-rank STS

Lemma (variant of [Jungnickel and Tonchev, 2018+])

Given a subspace D from $\dot{\mathcal{D}}_j$, the set of $STS(w-1)$ orthogonal to D is in one-to-one correspondence with the collections of one $STS(w/2^j-1)$, 2^j-1 symmetric latin squares of order $w/2^j-1$, and $(2^j-1)(2^j-2)/6$ latin squares of order $w/2^j$.

- triples with 3 ones in the "all-zero-column group" form to $\mathrm{STS}(w/2^j-1)$;
- for different columns a, b, c with $a+b+c=0$ we have a latin square (similarly to the 3-ary case);
- for non-zero column a the triples with two points in the corresponding group of coordinates and one point in the "all-zero group" form a symmetric square of order $w/2^j$.

The structure of non-full-2-rank STS

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Symmetric latin squares

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$$
\begin{array}{c}
0000000\ldots 11111111\ldots \\
0000000\ldots 00000000\ldots \\
0000000\ldots 11111111\ldots \\
1 \qquad\qquad 1 \qquad\qquad 1
\end{array}
$$

$$
\boxed{\begin{array}{cccc}
\mathbf{4} & 1 & 3 & \mathbf{2} \\
1 & \mathbf{4} & 2 & \mathbf{3} \\
3 & 2 & \mathbf{4} & \mathbf{1} \\
\mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{4}
\end{array}}
\longleftrightarrow
\boxed{\begin{array}{ccc}
\mathbf{2} & 1 & 3 \\
1 & \mathbf{3} & 2 \\
3 & 2 & \mathbf{1}
\end{array}}
$$

Remark. Symmetric latin squares of order $2u-1$ are in 1-to-1 correspondence with the (ordered) factorizations of the complete graph of order $2n$.

The rest of technique is the same as in the case of GF(3).

Main theorem, GF(2)

Theorem

Assume that w is divided by 2^k and k is the largest integer with this property. Let $i \in \{0, \ldots, k\}$. The total number of different $STS(w-1)$ of 2-rank $w-i-1$ equals

$$\dot{\Gamma}_{w,0,i} \sum_{j=i}^{k} \dot{\Gamma}_{w,i,j} \mu_{j-i}^{(2)} \dot{\Phi}_{w-1,j},$$

where $\mu_l^{(2)} = (-1)^l 2^{\binom{l}{2}}$, $\dot{\Phi}_{w-1,j} = \Psi_{w/2^j-1} \Pi_{w/2^j-1}^{2^j-1} \Lambda_{w/2^j}^{(2^j-1)(2^j-2)/6}$, Ψ_u is the number of $STS(u)$ (and also the number of idempotent totally symmetric latin squares of order u), Π_u is the number of symmetric latin squares of order u (and also $u!$ times the number of 1-factorizations of the complete graph of order $u+1$), Λ_u is the number of latin squares of order u, and

$$\dot{\Gamma}_{w,i,j} = \left(\frac{w}{2^i}!\right)^{2^i} \Big/ 2^{\frac{(j-i)(j+i+1)}{2}} \left(\frac{w}{2^j}!\right)^{2^j} \prod^{j-i} (2^s - 1)$$

Partial cases, 2-rank +1 and +2

Corollary ([Tonchev 2001])

The number of STS$(w-1)$, $w=2^k$, of 2-rank $w-k$ is

$$w!(2^{w^2/24-3w/4+k+1/3}-1)/2^{\frac{k(k+1)}{2}}[k-1]_2!$$

Corollary ([Zinoviev–Zinoviev 2013])

The number of STS$(w-1)$, $w=2^k$, of 2-rank $w-k+1$ is

$$\frac{w!\left(3^{w^2/48-w/4+2/3}\cdot 2^{w^2/16-5w/4+2k-1}-3\cdot 2^{w^2/24-3w/4+k-2/3}+1\right)}{3\cdot 2^{\frac{(k+2)(k-1)}{2}}\cdot [k-2]_2!}$$

Partial cases, 2-rank +3 and minimal for $w = 10 \cdot 2^k$

Corollary ([Zinoviev 2016])

The number of $STS(w-1)$, $w = 2^k$, *of 2-rank* $w - k + 2$ *is*

$$\frac{2^k!}{21 \cdot 2^{\frac{k(k+1)}{2}-3} \cdot [k-3]_2!}$$

$$\times \Big(780^{w/8-1} \cdot (2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 1361291)^{w^2/384 - w/16 + 1/3} \cdot 2^{3k-12}$$

$$-7 \cdot 2^{w^2/16 - 5w/4 + 2k - 3} \cdot 3^{w^2/48 - w/4 + 2/3} + 7 \cdot 2^{w^2/24 - 3w/4 - 5/3 +}$$

Corollary

The number of $STS(10w-1)$, $w = 2^k$, *of 2-rank* $10w - 1 - k$ *is*

$$\frac{(10w)! \cdot 122556672^{w-1} \cdot (2^{43} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 31 \cdot 37 \cdot 547135293937)^{\frac{(w-1)(w}{6}}}{2^{\frac{k(k+1)}{2}+5} \cdot 135 \cdot [k]_2!}$$

non-maximal 2-rank and 3-rank together?

Theorem

There is no a Steiner triple system of order v larger than 3 that is both non-full-2-rank and non-full-3-rank, i.e., of 2-rank less than v and 3-rank less than $v-1$.

Assume that S an a STS(v), $v > 3$, which is
(i) of 3-rank at most $v-2$ and
(ii) 2-rank at most $v-1$, $v > 3$.
By Lemma, (i) means that there is a vector with $v/3$ zeros, $v/3$ ones, and $v/3$ twos that is orthogonal to S over GF(3); in particular, $v \equiv 0 \bmod 3$. Assumption (ii) means that S has a Steiner subsystem S' of order $(v-1)/2$, by Lemma. Since $v > 3$ implies $(v-1)/2 > v/3$, the system S' is orthogonal over GF(3) to a vector that is not all-0, all-1, or all-2. By Lemma 1, the order $(v-1)/2$ is an integer divisible by 3, and we get $v \equiv 1 \bmod 3$, a contradiction.

THANK YOU !!!