Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra

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Abstract

Let $\Omega$ be a finite dimensional central algebra with an involutorial antiautomorphism $\sigma$ and $\text{char } \Omega \neq 2$. $\Omega^{n \times n}$ be the set of all $n \times n$ matrices over $\Omega$. $A = (a_{ij}) \in \Omega^{n \times n}$ is called bisymmetric if $a_{ij} = a_{n-i+1,n-j+1} = \sigma(a_{ji})$ and biskewsymmetric if $a_{ij} = -a_{n-i+1,n-j+1} = -\sigma(a_{ji})$. The following systems of generalized Sylvester equations over $\Omega[\lambda]$:

\begin{align}
A_1 X - Y B_1 &= C_1, \\
&\vdots \\
A_s X - Y B_s &= C_s, \\
A_1 X B_1 - C_1 X D_1 &= E_1, \\
&\vdots \\
A_s X B_s - C_s X D_s &= E_s,
\end{align}

are considered. Necessary and sufficient conditions are given for the existence of constant solutions with bi(skew)symmetric constrains to (I) and (II). As a particular case, auxiliary results dealing with the system of Sylvester equations are also presented.

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1. Introduction

Throughout this paper, we denote a finite dimensional central algebra with an involution $\sigma$ [25, p.112, Definition 1] over a field $\mathcal{F}$ and chart $\Omega \neq 2$, the set of all $m \times n$ matrices over $\Omega$ by $\Omega^{m \times n}$, the set of all $n \times n$ invertible matrices over $\Omega$ by $\text{GL}_n(\Omega)$, the set of all $m \times n$ matrices over $\Omega[\lambda]$ is denoted by $\Omega^{m \times n}[\lambda]$, the quaternion field over real number field by $\mathbb{Q}$, $i \times i$ identity matrix by $I_i$, the $n \times n$ permutation matrix whose elements along the southwest–northeast diagonal are ones and whose remaining elements are zeros by $V_n$. Note that $V_n^* = V_n^{-1} = V_n$.

Let $A = (a_{ij}) \in \Omega^{m \times n}$, $A^* = (\sigma(a_{ij})) \in \Omega^{n \times m}$, $A(\equiv) = (\sigma(a_{m-j+1,n-i+1})) \in \Omega^{m \times n}$, $A^\# = (a_{m-i+1,n-j+1}) \in \Omega^{m \times n}$. Then $A^s = V_n A^* V_m$, $A^\# = V_m A V_n$. $A$ is called (skew)selfconjugate if $A = A^s (-A^*)$, per(skeenselfconjugate if $A = A(\equiv) (-A(\equiv))$, centro(skeensymmetric if $A^\# = A(-A^\#)$.

Of the three matrix properties—(skew)selfconjugate, per(skeenselfconjugate, and centro(skeensymmetric—any two imply the third one. We call $A$ a bi(skeensymmetric matrix if it is (skew)selfconjugate and per(skeenselfconjugate (or centro(skeensymmetric) at the same time.

It is well known that when $\Omega$ is the real field, the perselfconjugate matrix is persymmetric matrix and bisymmetric matrix is symmetric centrobsymmetric matrix which is widely useful in applied mathematics such as information theory, linear system theory, linear estimation theory and numerical analysis theory and the others (e.g. [18–21]).

For matrix $A$, $B$ over $\Omega$, it is easy to verify that

\begin{align*}
(A^*)^s &= A, \\
(A^s)^{(s)} &= A, \\
(A^\#)^\# &= A, \\
(AB)^s &= B^* A^*, \\
(AB)^{(s)} &= B^{(s)} A^{(s)}, \\
(AB)^\# &= A^\# B^\#. 
\end{align*}

Define $(A^*)^{-1} = A^{-s}$, $(A^{(s)})^{-1} = A^{(-s)}$, $(A^\#)^{-1} = A^{-\#}$ if $A$ is invertible. Suppose $A$, $B \in \Omega^{p \times n}$, $C$, $D \in \Omega^{m \times m}$, then $(A$, $B$, $C$, $D)$ is called a regular matrix quadruple if there exists $\lambda \in F$ such that $A + \lambda B$, $C + \lambda D$ are invertible.

$(A$, $B)$ is called a regular matrix pencil if there exists $\lambda \in F$ such that $A + \lambda B$ is invertible. Accordingly $\lambda$ is called the regular coefficient. Many problems in systems...
and control theory require the solution of Sylvester’s matrix equation $AX - XB = C$ or its generalization $AX - YB = C$. Roth [12] gave necessary and sufficient conditions for the consistency of the two matrix equations, which were called Roth’s theorems on the equivalence and similarity of block diagonal matrices. Since Roth’s paper appeared in 1952, Roth’s theorems have been widely extended (e.g. [1–11]). The matrix equation $AXB - CXD = E$ appears in the study of perturbations of the generalized eigenvalue problem [13], in the numerical solution of implicit ordinary differential equations [14], and in stability problems for descriptor systems [15]. Wimmer etc. made excellent contributions to the matrix equation (e.g. [16,17]). Perturbation analysis of generalized Sylvester eigenspaces of matrix quadruples [24] leads to pairs of generalized Sylvester equations of the form

$$A_1X - YB_1 = C_1,$$
$$A_2X - YB_2 = C_2,$$

which has been studied in [10]. In 1985 Guralnick [1] proved that Roth’s theorems on the equivalence and similarity of block diagonal matrices hold for finite sets of matrices over a commutative ring.

In this paper, the following systems of matrix equations over $\mathbb{F}[\lambda]$:

$$A_1X - YB_1 = C_1,$$
$$\vdots$$
$$A_sX - YB_s = C_s,$$

$$A_1XB_1 - C_1XD_1 = E_1,$$
$$\vdots$$
$$A_sXB_s - C_sXD_s = E_s$$

are considered. Necessary and sufficient conditions are given for the existence of bi(skew)symmetric solutions over $\mathbb{F}$ to (1.1) and (1.2), a solution $(X, Y)$ over $\mathbb{F}$ which satisfies $X$ is bisymmetric (biskewsymmetric) and $Y$ is biskewsymmetric (bisymmetric) to (1.1). As a particular case, auxiliary results dealing with the system of Sylvester equations are also presented.

2. Main results

To begin with the following:

**Theorem 1.** Let $A_i, B_i, C_i \in \mathbb{F}^{m \times n}[\lambda]$. Then the system (1.1) has a bi-symmetric solution $(X, Y)$ over $\mathbb{F}$, i.e., $X^* = X = X^{(s)}$, $Y^* = Y = Y^{(s)}$, if and only if there exist $P \in GL_{2n}(\mathbb{F})$, $Q \in GL_{2m}(\mathbb{F})$ such that
Q\[\begin{bmatrix}A_i & C_i \\ O & B_i\end{bmatrix}\] = \[\begin{bmatrix}A_i & O \\ O & B_i\end{bmatrix}\] P, \hspace{1em} 1 \leq i \leq s, \hspace{1em} (2.1)

\[\begin{bmatrix}O & I_n \\ -I_n & O\end{bmatrix}\] P = \[\begin{bmatrix}O & I_n \\ -I_n & O\end{bmatrix}\], \hspace{1em} (2.2)

\[\begin{bmatrix}O & I_n \\ -I_n & 0\end{bmatrix}\] P\[\begin{pmatrix}0 & I_n \\ I_n & 0\end{pmatrix}\] = \[\begin{bmatrix}O & I_n \\ O & I_n\end{bmatrix}\], \hspace{1em} (2.3)

\[\begin{bmatrix}0 & I_n \\ I_n & O\end{bmatrix}\] \[\begin{pmatrix}0 & I_n \\ I_n & O\end{pmatrix}\] = \[\begin{bmatrix}O & I_n \\ I_n & O\end{bmatrix}\], \hspace{1em} (2.4)

\[\begin{bmatrix}O & I_m \\ -I_m & 0\end{bmatrix}\] Q^* = \[\begin{bmatrix}O & I_m \\ -I_m & O\end{bmatrix}\], \hspace{1em} (2.5)

\[\begin{bmatrix}O & I_m \\ -I_m & O\end{bmatrix}\] Q\[\begin{pmatrix}0 & I_m \\ O & I_m\end{pmatrix}\] = \[\begin{bmatrix}O & I_m \\ O & I_m\end{bmatrix}\], \hspace{1em} (2.6)

\[\begin{bmatrix}O & I_m \\ I_m & O\end{bmatrix}\] Q^{-1} = \[\begin{bmatrix}O & I_m \\ I_m & O\end{bmatrix}\], \hspace{1em} (2.7)

**Proof.** Suppose that

\[M_{i0} = \begin{bmatrix}A_i & O \\ O & B_i\end{bmatrix}, \hspace{1em} M_{ic} = \begin{bmatrix}A_i & C_i \\ O & B_i\end{bmatrix},\]

\[E_t = \begin{bmatrix}O & I_t \\ I_t & O\end{bmatrix}, \hspace{1em} F_t = \begin{bmatrix}O & 0 \\ I_t & O\end{bmatrix}, \hspace{1em} J_t = \begin{bmatrix}O & I_t \\ -I_t & O\end{bmatrix}.\]

Note that

\[E_t^* = E_t^{-1} = E_t^#, \hspace{1em} E_t^{(s)} = E_t,\]

\[J_t^* = J_t^{-1} = -J_t, \hspace{1em} J_t^{(s)} = J_t,\]

\[F_t^* = F_t, \hspace{1em} F_t^{(s)} = F_t^# = -F_t.\]

Let the system (1.1) have a bisymmetric solution \((X, Y)\) over \(\Omega\) and

\[P = \begin{bmatrix}I_n & X \\ O & I_n\end{bmatrix}, \hspace{1em} Q = \begin{bmatrix}I_m & Y \\ O & I_m\end{bmatrix}.\]

Then it follows from \(X^* = X = X^{(s)}, \hspace{1em} Y^* = Y = Y^{(s)}\) that \(X^# = X, \hspace{1em} Y^# = Y\). Hence it is easy to verify that (2.1)–(2.7) hold.

Conversely, suppose that (2.1)–(2.7) hold. Let

\[U = \begin{bmatrix}U_1 & U_{12} \\ U_{21} & U_2\end{bmatrix} \in \Omega^{2m \times 2m}, \hspace{1em} V = \begin{bmatrix}V_1 & V_{12} \\ V_{21} & V_2\end{bmatrix} \in \Omega^{2n \times 2n}.\]
Then
\[
U^\ast = \begin{bmatrix}
U_{11}^\ast & U_{21}^\ast \\
U_{12}^\ast & U_{22}^\ast
\end{bmatrix}^m \in \Omega^{2m \times 2m},
\]
\[
V^{(s)} = \begin{bmatrix}
V_{22}^{(s)} & V_{12}^{(s)} \\
V_{21}^{(s)} & V_{11}^{(s)}
\end{bmatrix}^n \in \Omega^{2n \times 2n},
\]
\[
U^\# = \begin{bmatrix}
U_{11}^\# & U_{21}^\# \\
U_{12}^\# & U_{22}^\#
\end{bmatrix}^m \in \Omega^{2m \times 2m}.
\]

Suppose that
\[
T_c = \{(U, V) \mid M_{0i}V = U M_{i0c}, \quad 1 \leq i \leq s\},
\] (2.8)
\[
S_c = \{(U, V) \mid J_{m}U^* J_{m}M_{0i} = M_{i0} J_{n}V^* J_{n}, \quad 1 \leq i \leq s\},
\] (2.9)
\[
K_c = \{(U, V) \mid F_{m}U^{(s)} F_{m}M_{i0} = M_{i0c} F_{n}V^{(s)} F_{n}, \quad 1 \leq i \leq s\},
\] (2.10)
\[
L_c = \{(U, V) \mid E_{m}U^\# E_{m}M_{ic} = M_{0i}E_{n}V^\# E_{n}, \quad 1 \leq i \leq s\}.
\] (2.11)

It is easy to verify that \(T_c, S_c, K_c, L_c\) are finite dimensional linear spaces over \(F\) with scalar multiplication and addition defined as follows:

\[(U, V)b = (Ub, Vb), \quad b \in F,\]
\[(U_1, V_1) + (U_2, V_2) = (U_1 + U_2, V_1 + V_2).\]

Then for \((U, V) \in T_c\), we have the following:
\[
A_i V_1 - U_1 A_i = 0, \\
B_i V_{21} - U_2 A_i = 0, \\
A_i V_{12} - U_1 C_i - U_{12} B_i = 0, \\
B_i V_2 - U_{21} C_i - U_2 B_i = 0,
\] (2.12)

The conditions for \((U, V) \in S_c\) are
\[
U_{21}^* A_i - A_i V_2^* + C_i V_{21}^* = 0, \\
A_i V_{12} - U_{12}^* B_i - C_i V_{11}^* = 0, \\
B_i V_{21}^* - U_{21}^* A_i = 0,
\] (2.13)
\((U, V) \in K_c\) is equivalent to the following:

\[
U_2^{(s)} A_i - A_i V_2^{(s)} + C_i V_{21}^{(s)} = 0,
A_i V_{12}^{(s)} - U_{12}^{(s)} B_i - C_i V_1^{(s)} = 0,
B_i V_{21}^{(s)} - U_{21}^{(s)} A_i = 0, \\
U_1^{(s)} B_i - B_i V_1^{(s)} = 0.
\] (2.14)

From \((U, V) \in L_c\), we have

\[
U#_1 A_i - A_i V#_1 = 0, \\
A_i V#_{12} - U#_{12} B_i - U#_1 C_i = 0, \\
B_i V#_{21} - U#_{21} A_i = 0, \\
U#_2 B_i - B_i V#_2 + U#_2 C_i = 0,
\] (2.15)

Put

\[
W_c = T_c \cap S_c \cap K_c \cap L_c. 
\] (2.16)

Clearly \(W_c\) is a finite dimensional linear space over \(F\). For \(C_i = 0, 1 \leq i \leq s\), let \(T_0, S_0, K_0, L_0\) and \(W_0\) be defined by (2.8), (2.9), (2.10), (2.11) and (2.16), respectively. By (2.1)–(2.11) and (2.16), we can verify that \((U, V) \in W_c \iff (UQ^{-1}, VS^{-1}) \in W_0\). Hence

\[
\dim W_c = \dim W_0. 
\] (2.17)

Define linear maps \(f : \Omega^{2m \times 2m} \times \Omega^{2n \times 2n} \to \Omega^{2m \times m}\) and \(g : \Omega^{2m \times 2m} \times \Omega^{2n \times 2n} \to \Omega^{2n \times n}\) by the following:

\[
f(U, V) = \begin{bmatrix} U_1 \\ U_{21} \end{bmatrix}, \quad g(U, V) = \begin{bmatrix} V_1 \\ V_{21} \end{bmatrix},
\]
respectively. It is obvious that in the case \(C = 0\) we have \((U, V) = (I_{2m}, I_{2n}) \in W_0\) and therefore

\[
\begin{bmatrix} I_m \\ 0 \end{bmatrix} \in f(W_0), \quad \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in g(W_0).
\] (2.18)

Let \(f_c = f|_{W_c}, \quad f_0 = f|_{W_0}, \quad g_c = g|_{W_c}, \quad g_0 = g|_{W_0}\). Then it follows from (2.12) to (2.15) that

\[
ker f_c = ker f_0, \quad ker g_c = ker g_0.
\] (2.19)

Suppose \(U\) and \(V\) are as above mentioned and

\[
G = \begin{bmatrix} U_1 \\ U_{21} \end{bmatrix}, \quad H = \begin{bmatrix} V_1 \\ V_{21} \end{bmatrix}.
\]

Then it follows from (2.12) to (2.15) that if \((U, V) \in W_c\) then \((G, H) \in W_0\). Hence we have

\[
\text{Im } f_c \subseteq \text{Im } f_0, \quad \text{Im } g_c \subseteq \text{Im } g_0.
\] (2.20)
According to (2.17), (2.19) and
\[ \dim \ker f_c + \dim \text{Im} f_c = \dim W_c, \quad \dim \ker f_0 + \dim \text{Im} f_0 = \dim W_0, \]
we have \( \dim \text{Im} f_c = \dim \text{Im} f_0, \dim \text{Im} g_c = \dim \text{Im} g_0. \) Therefore (2.20) yields
\[ \text{Im} f_c = \text{Im} f_0, \text{Im} g_c = \text{Im} g_0, \]
i.e., \( f(W_c) = f(W_0), g(W_c) = g(W_0). \) By (2.18),
\[ \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in f(W_c), \quad \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in g(W_c). \]
Hence there exists \((U, V) \in W_c\) such that
\[ U_{12} = \text{Im}, V_{12} = \text{In}. \]
In view of (2.12)–(2.15), we have
\[ \begin{bmatrix} A_i & V_{12} \\ 0 & U_{12} \end{bmatrix} - \begin{bmatrix} U_{12} & 0 \\ -I_{12} & I_{12} \end{bmatrix} B_i = C_i, \]
\[ \begin{bmatrix} A_i & V_{12} \\ 0 & U_{12} \end{bmatrix} - \begin{bmatrix} U_{12} & 0 \\ -I_{12} & I_{12} \end{bmatrix} B_i = C_i, \]
\[ \begin{bmatrix} A_i & V_{12} \\ 0 & U_{12} \end{bmatrix} - \begin{bmatrix} U_{12} & 0 \\ -I_{12} & I_{12} \end{bmatrix} B_i = C_i, \]
\[ \begin{bmatrix} A_i & V_{12} \\ 0 & U_{12} \end{bmatrix} - \begin{bmatrix} U_{12} & 0 \\ -I_{12} & I_{12} \end{bmatrix} B_i = C_i. \]

Let
\[ X = \frac{1}{4}(V_{12} + V_{12}^* + V_{12}^{(s)} + V_{12}^h), \]
\[ Y = \frac{1}{4}(U_{12} + U_{12}^* + U_{12}^{(s)} + U_{12}^h). \]
Then \( X^* = X = X^{(s)} = X^h, \ Y^* = Y = Y^{(s)} = Y^h \) and \((X, Y)\) is a solution of the system (1.1). \( \square \)

**Corollary 1.** Let \( A_i, B_i, C_i \in \Omega^{m \times n}, 1 \leq i \leq s. \) Then the system (1.1) has a bisymmetric solution \((X, Y)\) over \( \Omega, \) i.e., \( X^* = X = X^{(s)}, Y^* = Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2n}(\Omega), Q \in GL_{2m}(\Omega) \) such that (2.1)–(2.7) hold.

**Corollary 2.** Let \( A, B, C \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{m \times n} \)). Then the generalized Sylvester equation \( AX - YB = C \) has a bisymmetric solution \((X, Y)\) over \( \Omega, \) i.e., \( X^* = X = X^{(s)}, Y^* = Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2n}(\Omega), Q \in GL_{2m}(\Omega) \) such that
\[ Q \begin{bmatrix} A & C \\ O & B \end{bmatrix} P = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \]
and (2.2)–(2.7) hold.

**Remark 1.** This corollary which deals with constant solutions of matrix equation over \( \Omega[\lambda] \) may be of interest in its own right (see [23]).

**Theorem 2.** Let \( A_i, B_i, C_i \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{m \times n} \)), \( 1 \leq i \leq s. \) Then the system of Sylvester equations
\[ A_1 X - X B_1 = C_1, \]
\[ \vdots \]
\[ A_s X - X B_s = C_s. \]

has a bisymmetric solution \( X \) over \( \Omega \), i.e., \( X^* = X = X^{(s)} \), if and only if there exists \( P \in \text{GL}_{2n}(\Omega) \) such that

\[
\begin{bmatrix}
A_i & C_i \\
O & B_i
\end{bmatrix} = P^{-1} \begin{bmatrix}
A_i & O \\
O & B_i
\end{bmatrix} P, \quad 1 \leq i \leq s
\]

(2.25)

and (2.2)–(2.4) hold.

**Proof.** In the proof of Theorem 1, let \( m = n, V = U, P = Q, X = Y \) and adjust slightly the rest of the proof of Theorem 1. Then we can complete the proof. \( \square \)

**Corollary 3.** Let \( A, B, C \in \Omega^{n \times n} [\lambda] \) (or \( \Omega^{m \times n} \)). Then the Sylvester equation \( AX - XB = C \) has a bisymmetric solution \( X \) over \( \Omega \), i.e., \( X^* = X = X^{(s)} \), if and only if there exists \( P \in \text{GL}_{2n}(\Omega) \) such that

\[
\begin{bmatrix}
A & C \\
O & B
\end{bmatrix} = P^{-1} \begin{bmatrix}
A & O \\
O & B
\end{bmatrix} P
\]

(2.26)

and (2.2)–(2.4) hold.

**Theorem 3.** Let \( A_i, B_i, C_i \in \Omega^{m \times n} [\lambda], 1 \leq i \leq s \). Then the system (1.1) has a biskewsymmetric solution \( (X, Y) \) over \( \Omega \), i.e., \( X^* = -X = X^{(s)}, Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in \text{GL}_{2n}(\Omega), Q \in \text{GL}_{2m}(\Omega) \) such that (2.1) hold and

\[
p^{(s)} P = I_{2n}, \quad P \begin{bmatrix} O & I_m \\ I_n & O \end{bmatrix} P^* = \begin{bmatrix} O & I_m \\ I_n & O \end{bmatrix} = P^* \begin{bmatrix} O & I_m \\ I_n & O \end{bmatrix} P^{-1},
\]

(2.27)

\[
Q^{(s)} Q = I_{2m}, \quad Q \begin{bmatrix} O & I_m \\ I_m & O \end{bmatrix} Q^* = \begin{bmatrix} O & I_m \\ I_m & O \end{bmatrix} = Q^* \begin{bmatrix} O & I_m \\ I_m & O \end{bmatrix} Q^{-1}.
\]

(2.28)

**Proof.** In the proof of Theorem 1, replace \( S_c, K_c \) by

\[
S_c = \{(U, V) | E_n U^* E_n^0 = M_{1c} E_n V^{*0} E_n, 1 \leq i \leq s\},
\]

\[
K_c = \{(U, V) | U^{(*)} M_{1c7} V^{(*)}, 1 \leq i \leq s\},
\]

respectively. Then (2.13) and (2.14) become respectively the following

\[
U_i^* A_i - A_i^* V_i^* - C_i^* V_i^{*0} = 0,
\]

\[
A_i^* V_i^* - U_i^* B_i + C_i V_i^* = 0,
\]

\[
B_i^* V_i^{*0} - U_i^* A_i = 0,
\]

\[
U_i^* B_i - B_i^* V_i^* = 0.
\]
Corollary 4. Let \( A_i, B_i, C_i \in \Omega^{m \times n} \), \( 1 \leq i \leq s \). Then the system (1.1) has a biskewsymmetric solution \((X, Y)\) over \( \Omega \), i.e., \( X^* = -X = X^{(s)} \), \( Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2n}(\Omega) \), \( Q \in GL_{2m}(\Omega) \) such that (2.21), (2.27), (2.28) hold.

Corollary 5. Let \( A, B, C \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{n \times m} \)). Then the generalized Sylvester equation \( AX - YB = C \) has a biskewsymmetric solution \((X, Y)\) over \( \Omega \), i.e., \( X^* = -X = X^{(s)} \), \( Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2n}(\Omega) \), \( Q \in GL_{2m}(\Omega) \) such that (2.23), (2.27), (2.28) hold.

Theorem 4. Let \( A_i, B_i, C_i \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{n \times m} \)), \( 1 \leq i \leq s \). Then the system (2.24) has a biskewsymmetric solution \( X \) over \( \Omega \), i.e., \( X^* = -X = X^{(s)} \), if and only if there exists \( P \in GL_{2n}(\Omega) \) such that (2.25) and (2.27) hold.

Corollary 6. Let \( A, B, C \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{n \times m} \)). Then the Sylvester equation \( AX - XB = C \) has a biskewsymmetric solution \( X \) over \( \Omega \), i.e., \( X^* = -X = X^{(s)} \), if and only if there exists \( P \in GL_{2n}(\Omega) \) such that (2.26) and (2.27) hold.

Theorem 5. Let \( A_i, B_i, C_i \in \Omega^{m \times n}[\lambda] \), \( 1 \leq i \leq s \). Then the system (1.1) has a solution \((X, Y)\) over \( \Omega \) which satisfies that \( X \) is bisymmetric, i.e., \( X^* = X = X^{(s)} \), and \( Y \) is bisymmetric, i.e., \( Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2n}(\Omega) \). \( Q \in GL_{2m}(\Omega) \) such that (2.1) – (2.4) and (2.28) hold.

Proof. In the proof of Theorem 1, substitute \( S_c, K_c \) with

\[
S_c = \{(U, V)|E_m U^* E_m M_{10} = M_{1c} J_n V^* J_n^*, 1 \leq i \leq s\},
\]
and

\[ K_c = \{(U, V) | U^{(s)} M_{j0} = M_{ic} F_n V^{(s)} F_n, \; 1 \leq i \leq s\}, \]

respectively. Then (2.13) and (2.14) become respectively the following:

\[ U^*_2 A_i - A_i V^*_2 + C_i V^*_2 = 0, \]
\[ A_i V^*_2 + U^*_2 B_i - C_i V^*_i = 0, \;
1 \leq i \leq s, \]
\[ B_i V^*_2 + U^*_2 A_i = 0, \]
\[ U^*_i B_i - B_i V^*_i = 0. \]

Replacing (2.21), (2.22) with

\[ A_i V_{12} - U_{12} B_i = C_i, \]
\[ A_i V^*_2 = (-U^*_2) B_i = C_i, \]
\[ A_i V^*_2 = (-U^*_2) B_i = C_i, \]
\[ A_i V^*_2 - U^*_2 B_i = C_i, \]
\[ X = \frac{1}{2} (V_{12} + V^*_2 + V^{(s)} + V^*_2), \]
\[ Y = \frac{1}{2} (U_{12} - U^*_2 + U^{(s)} + U^*_2) \]

respectively, and adjusting slightly the rest of the proof of Theorem 1, we can complete the proof. □

**Corollary 7.** Let \( A_i, B_i, C_i \in \Omega^{m \times n}, \; 1 \leq i \leq s \). Then the system (1.1) has a solution \( (X, Y) \) over \( \Omega \) which satisfies that \( X \) is bisymmetric, i.e., \( X^* = X = X^{(s)} \), and \( Y \) is biskewsymmetric, i.e., \( Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2s}(\Omega) \), \( Q \in GL_{2n}(\Omega) \) such that (2.1)–(2.4) and (2.28) hold.

**Corollary 8.** Let \( A, B, C \in \Omega^{m \times n}[\lambda] \) (or \( \Omega^{m \times n} \)). Then the generalized Sylvester equation \( AX - YB = C \) has a solution \( (X, Y) \) over \( \Omega \) which satisfies that \( X \) is bisymmetric, i.e., \( X^* = X = X^{(s)} \), and \( Y \) is biskewsymmetric, i.e., \( Y^* = -Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2s}(\Omega) \), \( Q \in GL_{2n}(\Omega) \) such that (2.23), (2.2)–(2.4) and (2.28) hold.

**Theorem 6.** Let \( A_i, B_i, C_i \in \Omega^{m \times n}[\lambda], \; 1 \leq i \leq s \). Then the system (1.1) has a solution \( (X, Y) \) over \( \Omega \) which satisfies that \( X \) is biskewsymmetric, i.e., \( X^* = -X = X^{(s)} \), and \( Y \) is bisymmetric, i.e., \( Y^* = Y = Y^{(s)} \), if and only if there exist \( P \in GL_{2s}(\Omega) \), \( Q \in GL_{2n}(\Omega) \) such that (2.1), (2.5)–(2.7) and (2.27) hold.
Corollary 10. Let $A, B, C \in \mathbb{Q}^{m \times n}$. Then the system (1.2) has a solution $(X, Y)$ over $\mathbb{Q}$, i.e., $X^* = -X = X^{(s)}$, and $Y$ is bisymmetric, i.e., $Y^* = Y = Y^{(s)}$, if and only if there exist $P \in GL_{2n}(\mathbb{Q})$, $Q \in GL_{2m}(\mathbb{Q})$ such that (2.23), (2.5)–(2.7) and (2.27) hold.

Now we consider the system (1.2) where $A_i, C_i, B_i, D_i \in \mathbb{Q}^{m \times n}[\lambda_i]$, $(A_i, C_i, B_i, D_i)$ is a regular matrix quadruple. Then there exist $\lambda_i \in F$ such that $C_i + \lambda_i A_i$ and $B_i + \lambda_i D_i$ are invertible, $1 \leq i \leq s$. Let
\[\tilde{A}_i = (C_i + \lambda_i A_i)^{-1} A_i, \]
\[\tilde{D}_i = D_i (B_i + \lambda_i D_i)^{-1}, \]
\[\tilde{E}_i = (C_i + \lambda_i A_i)^{-1} E_i (B_i + \lambda_i D_i)^{-1}. \]

Then the system (1.2) is equivalent to the following:
\[\tilde{A}_1 X - X \tilde{D}_1 = \tilde{E}_1, \]
\[\vdots \]
\[\tilde{A}_s X - X \tilde{D}_s = \tilde{E}_s. \]

By Theorems 2 and 4, Corollaries 3 and 6, we have the following:

**Theorem 7.** Let \(A_i, C_i, B_i, D_i, E_i \in \mathbb{R}^{n \times n} \) (or \(\mathbb{C}^{n \times n} \)), \((A_i, C_i, B_i, D_i)\) is a regular matrix quadruple, \(1 \leq i \leq s\), and \(\tilde{A}_i, \tilde{D}_i, \tilde{E}_i\) are defined by (2.29). Then the system (1.2) has a bisymmetric solution \(X\) over \(\mathbb{R}^{n \times n}\), i.e., \(X^* = X = X^*\), if and only if there exists \(P \in GL_{2n}(\Omega)\) such that
\[P^{-1} \begin{bmatrix} \tilde{A}_i & \tilde{E}_i \\ O & \tilde{D}_i \end{bmatrix} P = \begin{bmatrix} \tilde{A}_i & O \\ O & \tilde{D}_i \end{bmatrix}, \quad 1 \leq i \leq s \quad (2.30)\]
and (2.2)–(2.4) hold;

(i) has a biskewsymmetric solution \(X\) over \(\Omega\), i.e., \(X^* = -X = X^*\), if and only if there exists \(P \in GL_{2n}(\Omega)\) such that (2.30) and (2.27) hold.

**Corollary 11.** Assume that \(A, C, B, D, E \in \mathbb{R}^{n \times n}[\lambda]\) (or \(\mathbb{C}^{n \times n} \)), and \((A, C, B, D)\) is a regular matrix quadruple, \(\lambda\) is the regular coefficient and
\[\tilde{A} = (C + \lambda A)^{-1} A, \quad \tilde{E} = (C + \lambda A)^{-1} E (B + \lambda D)^{-1}, \quad \tilde{D} = D (B + \lambda D)^{-1}.\]

Then the matrix equation \(AXB - CXD = E\)

(i) has a bisymmetric solution if and only if there exists \(P \in GL_{2n}(\Omega)\) such that
\[P^{-1} \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{D} \end{bmatrix} P = \begin{bmatrix} \tilde{A} & \tilde{E} \\ 0 & \tilde{D} \end{bmatrix} \quad (2.31)\]
and (2.2)–(2.4) hold;

(ii) has a biskewsymmetric solution if and only if there exists \(P \in GL_{2n}(\Omega)\) such that (2.31) and (2.27) hold.

In [22], Wimmer gave a necessary and sufficient conditions for the consistency of the matrix equation
\[X - AXB = C \quad (2.32)\]
over a field. By Corollary 11 we can consider the bi(skw)symmetric solution over \( \Omega \) of (2.32). By the way, according to [17, Corollary 2], for \( A, B, C, D \in \mathbb{Q}^{n \times n} \), it follows from \((I_n, A), (B, I_n)\) are regular matrix pencils that \((I_n, B, A, I_n)\) is a regular matrix quadruple. Hence there exists \( \lambda \in F \) such that \( I_n + \lambda A \) and \( B + \lambda I_n \) are all invertible. Define
\[
\tilde{A} = (I_n + \lambda A)^{-1} A, \\
\tilde{D} = (B + \lambda I_n)^{-1}, \\
\tilde{E} = -(I_n + \lambda A)^{-1} C (B + \lambda I_n)^{-1}.
\]

**Corollary 12.** Let \( A, B, C \in \mathbb{Q}^{n \times n} \), \( \tilde{A}, \tilde{D}, \tilde{E} \) are defined as (2.33). Then the matrix equation (2.32)

(i) has a bisymmetric solution if and only if there exists \( P \in GL_{2n}(\mathbb{Q}) \) such that (2.31) and (2.2)–(2.4) hold;

(ii) has a biskewsymmetric solution if and only if there exists \( P \in GL_{2n}(\mathbb{Q}) \) such that (2.31) and (2.27) hold.

**References**